Cohomology of Groups - Some results

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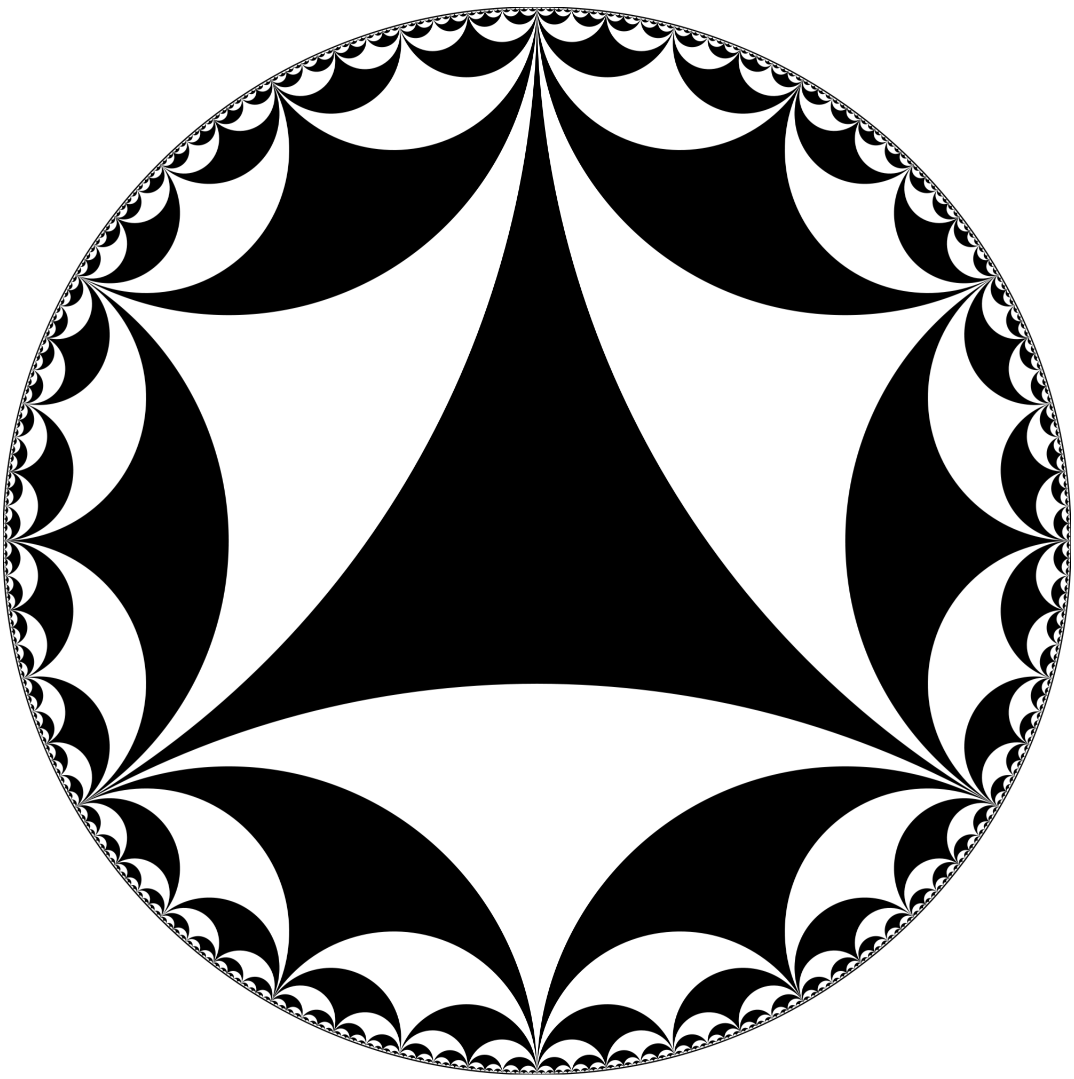


Contents

1 Some Homological Algebra

Proposition 1.0.1. *A chain complex C is contractible if and only if it is acyclic and each short exact sequence $0 \to Z_{n+1} \hookrightarrow C_{n+1} \xrightarrow{\bar{d}} Z_n \to 0$ splits, where $\bar{d}$ is induced by d.*

Proof. If h is a contracting homotopy, then $h_{|Z} \colon Z \to C$ splits the surjection $\bar{d} \colon C \to Z$. In fact we have $Id = hd + dh$, which became $Id = dh$ because we are restricted to Z, hence it's a section. Remember that by the acyclicity of the chain, $B = Z$. Conversely, suppose we have a splitting $s \colon Z \to C$, whence a graded module decomposition $C = \ker \bar{d} \oplus \operatorname{Im} s = Z \oplus \operatorname{Im} s$ (not a chain complex decomposition though). We then get a contracting homotopy $h \colon C \to C$ by setting

$$\begin{aligned} h \colon C &\to C \\ c_n &\mapsto \begin{cases} s_n(c_n) & \text{if } c_n \in Z_n \\ 0 & \text{else} \end{cases} \end{aligned}$$

notice that, any $c \in C_n$ can be written as $c = c_1 + s_{n-1}(c_2)$, where $c_1 \in Z_n$ and $c_2 \in Z_{n-1}$. So we have:

$$\begin{aligned} (hd + dh)(c) &= (hd + dh)(c_1 + s_{n-1}(c_2)) = hd(c_1 + s_{n-1}(c_2)) + dh(c_1 + s_{n-1}(c_2)) \\ &= hd(c_1) + hd(s_{n-1}(c_2)) + dh(c_1) + dh(s_{n-1}(c_2)) \\ &= ds_n(c_1) + h(c_2) \\ &= c_1 + s_{n-1}(c_2) = c \end{aligned}$$

□

Proposition 1.0.2. [1] *If $(C_\bullet, d)$ is an acyclic chain complex of projective $R-$modules and $C_\bullet$ is bounded below, then $C_\bullet$ is chain contractible.*

Proof. We construct a contraction $s_n \colon C_n \to C_{n+1}$ by induction on n to satisfy the needed condition

$$s_{n-1} \circ d_n + d_{n+1} \circ s_n = id_{C_n}$$

At the same time, we also show by induction that $\ker d_n$ is a direct summand in C_n. To begin the induction, set $s_j = 0$ for $j < 0$ and note that by assumption $H_0(C) = 0$ and $C_{-1} = 0$, hence $d_1 \colon C_1 \to C_0$ must be surjective. Since C_0 is projective, d_1 must have a right inverse s_0, so for $n = 0$ the base step holds. Furthermore, $\operatorname{Im} d_1 = \ker d_0 = C_0$ is projective (we use acyclicity here) and a direct summand of C_0 trivially.
For the inductive step, assume we've constructed s_j for $j < n$ to satisfy the step j^{th}, and we know $\ker d_j = \operatorname{Im} d_{j+1}$ is a direct summand in C_j for $j < n$ and hence projective. We shall construct s_n to satisfy the inductive step. By inductive assumption, $C_{n-1} = (\operatorname{Im} d_n) \oplus Q_{n-1}$ for some projective Q_{n-1}. On $\operatorname{Im} d_n = \ker d_{n-1}$, $s_{n-2} \circ d_{n-1} = 0$, so $d_n \circ s_{n-1}$ is the identity. Thus s_{n-1} is a right inverse for

$$d_n \colon C_n \to \operatorname{Im} d_n$$

Therefore $s_{n-1} \circ d_n$ is an idempotent endomorphism of C_n (i.e. $s_{n-1} \circ d_n \circ s_{n-1} \circ d_n = s_{n-1} \circ d_n$), with image Q_n complementary to $\ker d_n$, (easy to see that intersection is trivial and notice that $c = (c - s_{n-1} \circ d_n(c) + s_{n-1} \circ d_n(c)))$ and $\ker d_n = \operatorname{Im} d_{n+1}$ is R-projective. Since

$$d_{n+1} \colon C_{n+1} \to \operatorname{Im} d_{n+1} = \ker d_n$$

is surjective and the last module is projective, we have a section $s_n \colon \ker d_n \to C_{n+1}$. Extend s_n to all of C_n by making it 0 on Q_n. Then the step $j = n$ is satisfied and the proposition follows from induction. □

[1] From Algebraic K-Theory and Its Applications, of Jonathan Rosemberg, page 42

Proposition 1.0.3. *A chain map $f\colon (C_\bullet, d) \to (C'_\bullet, d')$ is a chain homotopy equivalence if and only if Cone(f) is contractible. If the complexes are bounded below and consist of projective R-modules, then it is a homotopy equivalence if and only if the mapping cone is acyclic, or if and only if it induces an isomorphism in homology.*

Proof. First observe that there is a s.e.s. of chain complexes

$$0 \to (C'_\bullet, d') \to (\text{Cone}(f), d'') \to (C[-1]_\bullet, -d) \to 0$$

The maps here are the obvious ones: we map $C'_j \to \text{Cone}(f)_j = C_{j-1} \oplus C'_j$ by the map $c' \mapsto (0, c')$ and we project $\text{Cone}(f)_j$ onto the first summand C_{j-1}. We have a l.e.s. in homology

$$\cdots \to H_n(\text{Cone}(f)) \to H_{n-1}(C) \xrightarrow{\partial} H_{n-1}(C') \to H_{n-1}(\text{Cone}(f)) \to \cdots$$

It's easy to see (diagram chasing in the snake lemma) that $\partial = f_*$. Thus f_* being an isomorphism in all degree is equivalent to the mapping cone being acyclic. Furthermore, if C and C'' are bounded below and consist of projective modules, then the same is true for Cone(f). Hence by Proposition 1.0.2, the mapping cone in this case is acyclic if and only if is contractible (other direction is trivial).

It remains to show that f is an homotopy equivalence if and only if Cone(f) is contractible. Suppose $s''\colon \text{Cone}(f) \to \text{Cone}(f)$ is a chain contraction. Then we define $s\colon C \to C$, $s'\colon C' \to C'$, and $\psi\colon C' \to C$ by

$$\begin{aligned} s''(c, 0) &= (s(c), \dots) \\ s''(0, c') &= (\psi(c'), -s'(c')) \end{aligned}$$

Since $d''s'' + s''d'' = Id_{\text{Cone}(f)}$, we have

$$\begin{aligned} (c, 0) &= (-d \circ s(c), \dots) + s''(-dc, f(c)) \\ &= (-d \circ s(c) + \psi \circ f(c) - s \circ d(c), \dots) \\ (0, c') &= d''(\psi(c'), -s'(c')) + s''(0, d'(c')) \\ &= (-d \circ \psi(c'), f \circ \psi(c') - d' \circ s'(c')) + (\psi \circ d'(c'), -s' \circ d'(c')) \end{aligned}$$

which says that ψ is a chain map, $\psi \circ f \overset{s}{\simeq} Id_C$ and $f \circ \psi \overset{s'}{\simeq} Id_{C'}$.

In the other direction, suppose f is a homotopy equivalence, with homotopy inverse ψ, and suppose one has homotopies s from $\psi \circ \varphi$ to Id_C and s' from $f \circ \psi$ to $Id_{C'}$. Let

$$\begin{aligned} s''(c, c') = (s(c) + \psi(c') + \psi \circ s' \circ f - \psi \circ f \circ s(c), \\ - s'(c') + s' \circ f \circ s(c) - (s')^2 \circ f(c)) \end{aligned}$$

It can be proved that this is the contraction we are looking for. □

Proposition 1.0.4. *Let $\mathcal{A}$ be an abelian category, a chain complex P is a projective object of $\boldsymbol{Ch}(\mathcal{A})$ if and only if, for every $i \in \mathbb{Z}$, P_i are projective object in $\mathcal{A}$, and the identity map $P \to P$ is null-homotopic. In particular, they must be exact.*

Proof. We prove now that a projective chain complex $P_\bullet$ is contractible. Let

$$0 \to P_\bullet \hookrightarrow \text{Cone}(Id) \xrightarrow{\pi} P_\bullet[-1]$$

Consider the following commutative triangle

$$\begin{tikzcd} & P_\bullet \arrow[dl, dotted, "\exists\psi"'] \arrow[d, "Id"] \\ \text{Cone}(Id)[1] \arrow[r, "\pi"] & P_\bullet \end{tikzcd}$$

Given by projectiviness of $P_\bullet$. By the fact that $\pi \circ \psi = \mathrm{Id}$, we have the following

$$\begin{aligned}\psi \colon P_{n-1} &\to P_{n-1} \oplus P_n \\ p &\mapsto (p, f(p))\end{aligned}$$

for some $f \colon P_{n-1} \to P_n$ morphism of modules. By the relation $\partial_{\mathrm{Cone}} \circ \psi = \psi \circ \partial_{P_\bullet[-1]}$ it's easy to see that f is the contraction we are looking for (recall that $\partial_{P_\bullet[-1]} = -\partial_P$ in order to let the projection being a chain map.). The fact that it is level-wise projective is trivial and therefore skipped.
Now assume $P_\bullet$ be a level-wise projective split exact complex (equivalent to contractible thanks to Prop 1.0.1), hence we have $P_n \simeq Z_n \oplus B'_n$, where we use the splitting s.e.s. $0 \to Z_n \to P_n \to \mathrm{coker} i_n \to 0$. We will denote $\mathrm{coker} i_n$ with B'_n. Note that B'_n and Z_n are projective modules, being direct summands of a projective. Define $d''_n \colon B'_n \to \mathrm{Im}(d_n)$ by restriction of d_n. It's clear that it's an isomorphism. We define the following complex $P(n) := \cdots \to 0 \to B'_n \to Z_{n-1} \to 0 \to \cdots$. Now $P_\bullet \simeq \bigoplus_{n \in \mathbb{Z}} P(n)$ by construction. Now let's consider the extension problem

$$\begin{tikzcd} & P_\bullet \arrow[d, "f"] \\ X_\bullet \arrow[r, "\pi"] & Y_\bullet \end{tikzcd}$$

f induces by restriction a morphism $f(n) \colon P(n) \to Y$ with $f = \sum_n f(n)$ (the sum is finite in each degree). It's easy to see that there is a morphism $g(n) \colon P(n) \to X$ with $\pi \circ g(n) = f(n)$. Hence $g := \bigoplus_n g(n) : P \to X$ satisfies $\pi \circ g = f$, and therefore $P_\bullet$ is projective. $\square$

Theorem 1.0.1. *Let $f \colon C' \to C$ be a weak equivalence between complexes of right R-modules. If P is a non-negative complex of flat left R-modules, then $f \otimes_R Id \colon C' \otimes_R P \to C \otimes_R P$ is a weak equivalence.*

Proof. Let $\mathrm{Cone}(f)$ be the mapping cone of f, it is acyclic by hypothesis. Notice now that $\mathrm{Cone}(f) \otimes_R P = \mathrm{Cone}(f \otimes_R Id)$, in fact degreewise they are the same and the differentials are seen to be equal after some computations. So it suffices to show that $\mathrm{Cone}(f) \otimes_R P$ is acyclic. Let $P^{(n)}$ be the truncation $(P_i)_{i \leq n}$. We will show inductively that $\mathrm{Cone}(f) \otimes_R P^{(n)}$ is acyclic. Note first that $\mathrm{Cone}(f) \otimes_R F$ is acyclic for any complex F consisting of a flat module concentrated in a single dimension, since the exact sequences $\mathrm{Cone}(f)_{i+1} \to \mathrm{Cone}(f)_i \to \mathrm{Cone}(f)_{i-1}$ remain exact when tensored with F. But $P^{(n)}/P^{(n-1)}$ is such a complex F. So if we assume inductively that $\mathrm{Cone}(f) \otimes_R P^{(n-1)}$ is acyclic, it follows from the exact sequence $0 \to \mathrm{Cone}(f) \otimes_R P^{(n-1)} \to \mathrm{Cone}(f) \otimes_R P^{(n)} \to \mathrm{Cone}(f) \otimes_R (P^{(n)}/P^{(n-1)}) \to 0$ that $\mathrm{Cone}(f) \otimes_R P^{(n)}$ is acyclic. Finally, $\mathrm{Cone}(f) \otimes_R P$ is the increasing union of acyclic complexes $\mathrm{Cone}(f) \otimes_R P^{(n)}$, hence it is acyclic. $\square$

Definition 1.0.1. We say that (G, A) is a G-module if G is a group, A is an abelian group together with an action of $G \curvearrowright A$ compatible with the abelian structure of A.

Note that A is a G-module if and only if A is a $\mathbb{Z}G$ module (with the classical algebraic definition of module).
We now show a very useful trick which we will use later in these notes. Let X be a G-set (i.e. a set with a G-action), then one forms the free abelian group $\mathbb{Z}[X]$ generated by X and one extends the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}[X]$. The resulting G-module is called a *permutation module*. In particular, one has a permutation module $\mathbb{Z}[G/H]$ for every subgroup H of G, where G/H is the set of cosets gH and G acts on G/H by left translation.
The operation of disjoint union in the category of G-sets corresponds to the direct sum operation in the category of G-modules:

$$\mathbb{Z}\left[\bigsqcup X_i\right] \approx \bigoplus \mathbb{Z} X_i$$

It follows that every permutation module $\mathbb{Z}[X]$ admits a decomposition

$$\mathbb{Z}[X] \approx \mathbb{Z}\left[\bigsqcup G/G_x\right] \approx \bigoplus \mathbb{Z}[G/G_x]$$

where x ranges over a set of representatives for the G-orbits in X and G_x is the isotropy subgroup of G at x. In particular, if X is a free G-set, (i.e. if all isotropy groups are trivial) then $G/G_x \cong G$ and we obtain:

Proposition 1.0.5. *Let X be a free G-set and let E be a set of representatives for the G-orbits in X. Then $\mathbb{Z}X$ is a free $\mathbb{Z}G$-module with basis E.*

Proof. Write $X = \bigsqcup_{x\in E} G/G_x$, and notice that $G/G_x \approx G$. Then $\mathbb{Z}[X] \approx \bigoplus_{x\in E} \mathbb{Z}[G]$ □

From now on, let C, C' be two chain complexes. We will denote the abelian group of homotopy classes of chain maps $C \to C'$ by $[C, C']$. It is often useful to interpret $[C, C']$ as the $0-th$ homology group of a certain *function complex* $\mathscr{H}_R(C, C')$, defined as follows: $\mathscr{H}_R(C, C')_n$ is the set of graded module homomorphisms of degree n from C to C', i.e. $\mathscr{H}_R(C, C')_n = \coprod_{q\in\mathbb{Z}} \hom_R(C_q, C'_{q+n})]$, and the boundary operator $\partial_n \colon \mathscr{H}_r(C, C')_n \to \mathscr{H}_R(C, C')_{n-1}$ is defined by $\partial_n(f) = \partial_{C'} \circ f - (-1)^n f \circ \partial_C$.
Note that 0-cycles are precisely the chain maps $C \to C'$, and the 0-boundaries are the null-homotopic chain maps. Thus $H_0(\mathscr{H}_R(C, C')) = [C, C']$

1.1 Some useful results

Proposition 1.1.1. *Let $f\colon K \to L$ be a chain map between chain complexes which consist of free modules over a principal ideal domain R. If f induces isomorphisms $f_*\colon H_*(K) \simeq H_*(L)$, then f is a chain equivalence.*

Proof. The exact homology sequence and the hypothesis imply that Cone(f) is acyclic. A submodule of a free R-module is free. hence the boundary groups of the complex Cone(f) are free, and therefore the exact sequence $0 \to \mathbb{Z}_n \to \text{Cone}(f)_n \to B_{n-1} \to 0$ splits (here we use aciclicity of the cone). Now we apply Proposition 1.0.3 and Proposition 1.0.1 to get the result. □

Proposition 1.1.2. *Let $f\colon C \to D$ be a chain map between complexes of free abelian groups. Suppose that for each field $\mathbb{F}$ the map $f \otimes \mathbb{F}$ induces isomorphisms of homology groups. Then f is a chain equivalence*

Proof. Let Cone(f) be the mapping cone of f. The hypothesis implies that $H_*(\text{Cone}(f)\otimes\mathbb{F}) = 0$ (We use the identification $\text{Cone}(f \otimes \mathbb{F}) = \text{Cone}(f) \otimes \mathbb{F}$ seen in Theorem 1.0.1). We use then the Universal Coefficient Sequence: it implies that $\text{Tor}(H_*(\text{Cone}(f), \mathbb{Z}/p) = 0$ for each prime p. Hence $H_*(\text{Cone}(f))$ is torsion-free (recall that $\text{Tor.}(H_*(X), \mathbb{Z}_p) \cong \ker(H_*(X) \xrightarrow{\cdot p} H_*(X))$ using balancedness). Now from the fact that $H_*(\text{Cone}(f) \otimes \mathbb{Q}) \simeq H_*(\text{Cone}(f)) \otimes \mathbb{Q}$, we have that $H_*(\text{Cone}(f)) \otimes \mathbb{Q} = 0$, which means that $H_*(\text{Cone}(f))$ is a torsion group. Hence the only possibility is that $H_*(\text{Cone}(f)) = 0$, and hence f is a quasi isomorphism, now we conclude with Proposition 1.1.1. □

2 Homology with $\mathbb{Z}$-coefficients

2.1 Coinvariants

If G is a group and M a G-module, then the group of coinvariants of M, denoted M_G, is defined to be the quotient of M by the additive subgroup generated by elements of the form

$g.m - m$ ($g \in G, m \in M$). We denote with M^G instead, the group of invariants, i.e. the largest submodule of M on which G acts trivially. Another description of M_G is given by

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}} M$$

where we regard $\mathbb{Z}$ as the trivial right $\mathbb{Z}G$-module. In view of the above equality, we have the following properties

1. Right-exactness: Given an exact sequence $M' \to M \to M'' \to 0$ of G-modules, the induced sequence $M'_G \to M_G \to M''_G \to 0$ is exact

2. If F is a free $\mathbb{Z}G$-module with basis (e_i), then F_G is a free Z-module with basis $([e_i])$.

Proposition 2.1.1. *Let X be a free G-complex and let Y be the orbit complex X/G. Then $C_*(Y) \cong C_*(X)_G$*

Proof. The projection $C_*(X) \to C_*(Y)$ induces, by passage to the quotient, a map $\varphi\colon C_*(X)_G \to C_*(Y)$. Now $C_*(X)_G$ has a $\mathbb{Z}$-basis with one basis element for each G-orbit of cells of X, and it is clear that φ maps a basis element of $C_*(X)_G$ to the corresponding basis element $C_*(Y)$ hence φ is an isomorphism ☐

2.2 The definition of H_*G

Definition 2.2.1. Let G be a group and $\epsilon\colon P \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. We define the homology groups of G by

$$H_iG := H_i(F_G)$$

Clearly such choice is independent of the choice of resolution, up to canonical isomorphism.

Example 2.2.1. Let $G \cong \mathbb{Z}/n\mathbb{Z}$, consider the resolution

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \to \mathbb{Z} \to 0$$

we obtain for P_G the complex

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

using the fact that $t.1 = t$ ($G \curvearrowright \mathbb{Z}G$ is defined from group multiplication) Thus

$$H_iG \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_n & i \text{ odd} \\ 0 & i \text{ even } i > 0 \end{cases}$$

2.3 Topological Interpretation

We begin with some standard definitions which naturally arise in this setting.

Definition 2.3.1. By G-complex we will mean a CW-complex X together with an action of G on X which permutes the cells. In other words we have, for each $g \in G$, and homeomorphism $g\colon X \to X$ which sends $x \mapsto g(x)$, such that the image $g(\sigma)$ of any cell σ of X is again a cell.

If X is a G-complex, then the action of G on X induces an action of G on the cellular chain complex $C_*(X)$ which thereby becomes a chain complex of G-modules. Moreover the canonical augmentation $\epsilon\colon C_0(X) \to \mathbb{Z}$ (defined by $\epsilon(v) = 1$ for every 0-cell v of X) is a map of G-modules.

Definition 2.3.2. We will say that X is a free G-complex if the action of G freely permutes the cells of X (i.e. $g\sigma \neq \sigma$ for all σ if $g \neq 1$).

In this case each chain module $C_n(X)$ has a $\mathbb{Z}$-basis which is freely permuted by G hence, by Proposition 1.0.5, $C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. Finally, if X is contractible, then $H_*(X) \approx H_*(\text{pt.})$; in other words, the sequence

$$\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \to \cdots \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is exact. We have, therefore:

Proposition 2.3.1. *Let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.*

Recall that Y is a $K(G,1)$ complex if

1. Y is connected
2. $\pi_1 Y = G$
3. The universal cover $\tilde{Y}$ of Y is contractible

Proposition 2.3.2. *If Y is a $K(G,1)$-complex then $H_*G \cong H_*Y$*

Proof. If Y is a $K(G,1)$-complex with universal cover $\tilde{Y}$, then we know that $C_*(\tilde{Y})$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, because G acts freely on it ($\pi_1 Y \curvearrowright \tilde{Y}$ by Deck Transformations). Since $C_*(\tilde{Y})_G \cong C_*(Y)$ by Prop 2.1.1, we proved the claim. □

2.4 Functoriality

Given a homomorphism $\alpha\colon G \to G'$ and a projective resolutions F and F' of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$, respectively, we can regard F' as a complex of G-modules via α. Then F' is clearly acyclic (although not projective, in general, over $\mathbb{Z}G$), so the fundamental lemma (existence and uniqueness of the lifting) gives rise to an augmentation-preserving G-chain map $\tau\colon F \to F'$, well-defined up to homotopy. The condition that τ be a G-map is expressed by the formula

$$\tau(g.x) = \alpha(g).\tau(x)$$

where we recall that the new action $G \curvearrowright F'$ is defined via the map α.
Clearly τ induces a map $F_G \to F'_{G'}$ because $\tau(g.m - m) = \tau(g.m) - \tau(m) = \alpha(g).\tau(m) - \tau(m)$, well defined up to homotopy, hence we obtain a well-defined map $\alpha_*\colon H_*G \to H_*G'$.

Proposition 2.4.1. *Fix $g_0 \in G$ and let $\alpha\colon G \to G$ be given by $g \mapsto g_0 g g_0^{-1}$. Then $\alpha_*\colon H_*G \to H_*G$ is the identity.*

Proof. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and define $\tau\colon F \to F$ by $x \mapsto g_0.x$. Clearly τ commutes with the boundary operators and satisfy the relation $\tau(g.x) = \alpha(g)\tau(x)$, so we can use this map to compute α_*. Notice now that τ induces the identity on F_G, because $\tau([x]) = [g_0.x] = [x]$, whence we have proved the claim. □

Corollary 2.4.1. *If G is a group and N is a normal subgroup, then the conjugation action of G on N induces an action of G/N on H_*N.*

Proof. First we show how G acts on $H_*(N)$.
Fix $g \in G$, we know that $G \curvearrowright N$ by conjugation, and to see how this action is transposed to the homology we do the following: consider a projective resolution of $\mathbb{Z}$ as an N-module (we will denote the action with $z.x$) and we equip the lower row of the following diagram with the new action $h \bullet x := ghg^{-1}.x$ induced by the conjugation action of G (this is the way to induce the action, as the definition shows).

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & (F_2, .) & \longrightarrow & (F_1, .) & \longrightarrow & (F_0, .) & \longrightarrow & (\mathbb{Z}, .) \\
 & & \downarrow{\scriptstyle \tau_2^g} & & \downarrow{\scriptstyle \tau_1^g} & & \downarrow{\scriptstyle \tau_0^g} & & \downarrow{\scriptstyle \tau_{-1}^g} \\
\cdots & \longrightarrow & (F_2, \bullet) & \longrightarrow & (F_1, \bullet) & \longrightarrow & (F_0, \bullet) & \longrightarrow & (\mathbb{Z}, \bullet)
\end{array}$$

The maps τ_n^g are simply multiplication by g. Everything commutes and every map involved is G equivariant, so we can take the map induced in homology to do our computations. As a side note, we took $\mathbb{Z} \xrightarrow{\tau_{-1}^g} \mathbb{Z}$ as a convention, because we need a G equivariant map with the action induced by conjugation. (Actually we could have used the identity, because due to the fact that the action of G on $\mathbb{Z}$ is trivial, multiplying by $g \in G$ doens't do anything, but we want to make this passage because it'll be easier to generalize to any G-module with possibly non trivial action)
It's easy now to see that the assignment $g \mapsto \tau_n^g$ is an action of G on $H_n(N)$ (it's invertible b/c we lifted an invertible function).
Now if we restrict to $h \in N \triangleleft G$, the map τ_n^h is the map induced by conjugation by an element of the group (the group now is N, because we are working in $H_*(N)$), and hence by Proposition 2.4.1 the action is trivial. Therefore we can factorize the action through G/N and we are done. $\square$

3 (co)Homology with any coefficients

3.1 Preliminaries on $\otimes_G$ and $\hom_G$

Recall that the tensor product $M \otimes_R N$ is defined whenever M is a right R-module and N is a left R-module. It is the quotient of $M \otimes_{\mathbb{Z}} N$ (we drop the $\mathbb{Z}$ symbol from now on) obtained by introducing the relations $m.r \otimes n = m \otimes r.n$. In case R is a group ring $\mathbb{Z}G$, we can avoid having to consider both left and right module by using the anti-automorphism $g \to g^{-1}$ of G. Thus we can regard any left G-module M as a right G-module by setting $m.g := g^{-1}.m$, and in this way we can make sense of the tensor product $M \otimes_G N$ of two left G-modules. Note that $M \otimes_G N$ is obtained from $M \otimes N$ by introducing the relations $g^{-1}.m \otimes n = m \otimes g.n$ (Which stands for the well known $m.g \otimes g.n$. Now notice that

$$m \otimes n = g^{-1}g.m \otimes n = g.m \otimes g.n$$

and we see that

$$M \otimes_G N = (M \otimes N)_G$$

where G acts *diagonally* on $M \otimes N$: $g.(m \otimes n) = g.m \otimes g.n$ and recall that $(M \otimes N)_G$ are the co-invariants. In particular this shows that $- \otimes_G -$ is commutative.
The diagonal G-action used above is quite general and can be used whenever a functor of one or several abelian groups is applied to G-modules. Consider, for example, the functor $\hom(\ , \ ) = \hom_{\mathbb{Z}}(\ , \ )$. If M and N are G-modules, then the action of G on M and N induces by functoriality a *diagonal* action of G on $\hom(M, N)$ given by

$$(g \bullet f) := g \circ f \circ g^{-1}$$

for $g \in G$, $f \colon M \to N$, $m \in M$ (by a little abuse of notation, we indicate with g both the element and the automorphism associated to it). The use of g^{-1} here is needed because of the contravariance of hom in the first variable. In effect, we compensate for the contravariance by converting M to a right-module. In fact, to being an action, it must satisfies the relation $(gh) \bullet f = g \bullet (h \bullet f)$, so we have

$$\begin{aligned}
((gh) \bullet f) &= (gh) \circ f \circ (gh)^{-1} \\
&= g \circ h \circ f \circ h^{-1} \circ g^{-1} \\
&= g \bullet (h \bullet f)
\end{aligned}$$

Note that $g \bullet f = f$ if and only if f commutes with the action of g. In fact

$$\begin{aligned} g \bullet f &= f \\ g \circ f \circ g^{-1} &= f \\ g \circ f &= f \circ g \end{aligned}$$

which leads to the following equality:

$$\hom_G(M, N) = \hom(M, N)^G$$

because, if $f \in \hom_G(M, N)$, then $f(g.m) = g.f(m)$, which is exactly $f \circ g = g \circ f$ (g acts on $f(m)$) and therefore by the above computation this is equivalent to $g \bullet f = f$.

3.2 Definition of $H_*(G, M)$ and $H^*(G, M)$

Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ and let M be a G-module.

Definition 3.2.1. We define the homology of G with coefficients in M by

$$H_*(G, M) = H_*(F \otimes_G M)$$

Here $F \otimes_G M$ can be thought of as the complex obtained from F by applying the functor $- \otimes_G M$. Thus Definition 3.2.1 is a natural generalization of the definition of H_*G we gave before. We can compute it even choosing projective resolutions of both $\mathbb{Z}$ and M, say $\epsilon\colon F \to \mathbb{Z}$ and $\eta\colon P \to M$, and setting

$$H_*(G, M) := H_*(F \otimes_G P) \tag{1}$$

Fortunately (1) is consistent with Definition 3.2.1; for η induces a weak equivalence $\mathrm{Id} \otimes \eta\colon F \otimes_G P \to F \otimes_G M$ by Theorem 1.0.1. Note that the same theorem gives us a weak equivalence $\epsilon \otimes \mathrm{Id}\colon F \otimes_G P \to \mathbb{Z} \otimes_G P$. Thus

$$H_*(G, M) = H_*(P_G) \tag{2}$$

Clearly, then, we have considerable flexibility in the choice of a chain complex from which to compute $H_*(G, M)$. For the moment we content ourselves with a trivial example of such computation:

$$H_0(G, M) \approx M_G$$

This follows from the right exactness of the tensor product. [Apply $-\otimes_G M$ to $F_1 \to F_0 \to \mathbb{Z} \to 0$ and use the Definition 3.2.1.
We turn now to *cohomology with coefficients*, which is defined via $\mathscr{H}$ rather than $\otimes$. Choose a projective resolution $F \to \mathbb{Z}$ as above and consider the complex $\mathscr{H}_G(F, M)$, where M is again regarded as a chain complex concentrated in dimension 0. Recall the definition of $\mathscr{H}_G$ we gave at the beginning of the notes; we have $\mathscr{H}_G(F, M)_n = \hom_G(F_{-n}, M)$. It is therefore reasonable to regard $\mathscr{H}_G(F, M)$ as a cochain complex by the usual indexing conventions, i.e. by setting

$$\mathscr{H}_G(F, M)^n = \mathscr{H}_G(F, M)_n = \hom_G(F_n, M)$$

It is then a non-negative cochain complex with coboundary operator δ given by

$$(\delta u)(x) := (-1)^{n+1} u(\partial x)$$

for $u \in \mathscr{H}_G(F, M)^n$, $x \in F_{n+1}$. We now define

$$H^*(G, M) = H^*(\mathscr{H}_G(F, M))$$

Remark 3.2.1. Because of the sign in the definition of the differential above, $\mathscr{H}_G(F, M)$ is not the same as the complex $\hom_G(F, M)$ obtained from F by applying the contravariant functor $\hom_G(-, M)$ dimension-wise. This clearly doesn't change cocycles, coboundaries or cohomology.

Note that the exact sequence $F_1 \to F_0 \to \mathbb{Z} \to 0$ yields an exact sequence $0 \to \hom_G(\mathbb{Z}, M) \to \hom_G(F_0, M) \to \hom_G(F_1, M)$. Since $\hom_G(\mathbb{Z}, M) = M^G$, this gives

$$H^0(G, M) = M^G$$

There exist analogues for $H^*(G, -)$ of 1 and 2, but these involve the notion of *injective resolution.*

3.3 (co)Restriction, (co)Extension, (co)Inflation

3.3.1 Restriction of Scalars

Let $\alpha\colon R \to S$ be a ring homomorphism. Then any S-module M, can be regarded as an R-module via α (i.e. $r \bullet m := \alpha(r).m$) and we obtain in this way a functor E_α from S-module to R-modules, called the restriction of scalars.

Proposition 3.3.1. *The assignment $E_\alpha\colon S\text{-}Mod \to R\text{-}Mod$ defined above is a functor*

Proof. As explained above, for any S-module M, $E(M)$ is a right module, via the action of R through α. Take a map of S-modules $g\colon M \to N$, and define $E(g)$ as simply g. We check now that g is still a map of R-modules $E(M) \to E(N)$:

$$g(r \bullet m) = g(\alpha(r).m) = \alpha(r).g(m) = r \bullet g(m)$$

therefore the map has the claimed properties.
Other properties are trivial seen to be true. $\square$

In a similar fashion, we speak about restriction in the context of G-modules, where G is a group:

Proposition 3.3.2. *If $\rho\colon H \to G$ is a group map, the (forgetful) functor E_ρ from G-modules to H-modules is exact. For every G-module A, there is a natural surjection $(E_\rho A)_H \to A_G$ and a natural injection $A^G \to (E_\rho A)^H$. These two maps extend uniquely to the two morphisms*

$$\textit{(corestriction) } cor_H^G\colon H_*(H; E_\rho A) \to H_*(G; A)$$
$$\textit{(restriction) } res_H^G\colon H^*(G; A) \to H^*(H, E_\rho A)$$

Proof. Exactness is very easy to see, as said above, the functor E_ρ doesn't do anything to maps. Both the natural surjection and injection are induced by the Identity on A. The statement about extension and uniqueness is a direct consequence of the properties of universal δ-functors, once we notice that the functors $T_*(A) := H_*(H; E_\rho A)$ and $T^*(A) := H^*(H, E_\rho A)$ are δ-functors (see Weibel, 2.1.4). $\square$

Remark 3.3.1. The terms restriction and corestriction are normally used only when H is a subgroup of G. In this case $\mathbb{Z}G$ is actually a free $\mathbb{Z}H$-module. Therefore every projective G-module is also a projective H-module. If A is a G-module, we may calculate cor_H^G as the homology $H_*(\alpha)$ of the chain map $\alpha\colon P \otimes_H A \to P \otimes_G A$. Similarly we may calculate res_H^G as the cohomology $H^*(\beta)$ of the map $\beta\colon \hom_G(P, A) \to \hom_H(P, A)$. This comes from the properties of homological δ-functors, which states that such maps are constructed in this precise way.

3.3.2 Functoriality of H_* and H^*

We state now precisely what is meant by functoriality of H_*. Have in mind that we have already defined it for $\mathbb{Z}$ coefficients.
Let $\mathcal{C}$ be the category of pairs (G, A), where G is a group and A is a G-module. A morphism in $\mathcal{C}$ from (H, B) to (G, A) is a pair $(\rho\colon H \to G, \varphi\colon B \to E_\rho A)$, where ρ is a group homomorphism and φ is an H-module map. Such a pair (ρ, φ) induces a map

$$\mathrm{cor}_H^G \circ \varphi\colon H_*(H, B) \to H_*(G, A)$$

It follows that H_* is a covariant functor form $\mathcal{C}$ to **Ab**.
Similarly, we state now precisely what is meant by functoriality of H^*.
Let $\mathcal{D}$ be the category with same object as $\mathcal{C}$, but a morphism in $\mathcal{C}$ from (H, B) to (G, A) is a pair $(\rho\colon H \to G, \varphi\colon E_\rho B \to A)$ (note the reverse direction of φ!). Such a pair (ρ, φ) induces a map

$$\varphi \circ \mathrm{res}_H^G\colon H^*(G, A) \to H^*(H, B)$$

It follows that H^* is a contravariant functor form $\mathcal{D}$ to **Ab**.

Example 3.3.1 (Conjugation). Suppose H is a subgroup of G, so that each $g \in G$ induces an isomorphism ρ between H and its conjugates gHg^{-1}. If A is a G-module, the abelian group map $\mu_g\colon A \to A$ $(a \mapsto g.a)$ is actually an H-module map from A to $E_\rho A$ because $\mu_g(h.a) = gh.a = (ghg^{-1})g.a = \rho(h).\mu_g(a)$, for all $h \in H$ and $a \in A$. In the category $\mathcal{C}$ defined above, (ρ, μ_g) is an isomorphism $(H, A) \simeq (gHg^{-1}, A)$. Similarly, (ρ, μ_g^{-1}) is an isomorphism $(H, A) \simeq (gHg^{-1}, A)$ in $\mathcal{D}$. Therefore we have maps $H_*(H; A) \to H_*(gHg^{-1}, A)$ and $H^*(gHg^{-1}; A) \to H^*(H; A)$. The way to compute these maps is the same as we showed for the case $A = \mathbb{Z}$.

3.3.3 Extension of Scalars

The purpose of this section is to study two constructions which go in the opposite direction, from $R-$modules to S-modules.
For any left R-module M, consider the tensor product $S \otimes_R M$, where S is regarded as a right R-module by $s \bullet r := s.\alpha(r)$. Since the natural left action of S on itself commutes with this right action of R on S i.e. S is a $S - R$-bimodule (use associativity of ring multiplication), we can make $S \otimes_R M$ a left S-module by setting

$$s(s' \otimes m) := s.s' \otimes m$$

This S-module is said to be obtained from M by **extension of scalars** from R to S.
Note that there is a natural map $i\colon M \to S \otimes_R M$ given by $i(m) = 1 \otimes m$. Since

$$1 \otimes r.m = 1 \bullet r \otimes m = \alpha(r) \otimes m = \alpha(r)(1 \otimes m)$$

for $r \in R$, we have

$$i(r.m) = \alpha(r).i(m)$$

In other words, i is an R-module map, where the S-module $S \otimes_R M$ is regarded as an R-module by restriction of scalars. Moreover, the following *universal mapping property* holds:

Proposition 3.3.3 (Universal Mapping Property)**.** *Given an S-module N and an R-module map $f\colon M \to N$, there is a unique S-module map $g\colon S \otimes_R M \to N$ such that $g \circ i = f$:*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"] & S \otimes_R M \arrow[dl, dotted, "g"] \\ N & \end{tikzcd}$$

Thus we have

$$\hom_S(S \otimes_R M, N) \approx \hom_R(M, E_\alpha(N))$$

Proposition 3.3.3 shows that the extension of scalars functor (R-Mod) $\to$ (S-Mod) is left adjoint to the restriction of scalars functor (S-Mod) $\to$ (R-Mod).
Heuristically, we have just showed that $S \otimes_R M$ is the smallest S-module which receives an R-module map from M.

Proof. We only need to note that g, if it exists, must satisfy $g(s \otimes m) = s.g(1 \otimes m) = s.g(i(m)) = s.f(m)$. This proves uniqueness and tells us how to define g in order to prove existence: it's enough to set $g(s \otimes m) := s.f(m)$. It's easy to see that g is a map of S-modules. □

Note that Prop 3.3.3 can be applied in particular,with $M = N$ (regarded as an R-module by restriction of scalars) and $f = Id_N$. We obtain, then, for any S-module N, a canonical S-module map

$$S \otimes_R N \to N$$

given by $s \otimes n \mapsto s.n$. This map is surjective; moreover as an R-module map, it is a split surjection.

3.3.4 Co-extension of Scalars

We now consider a dual construction , which uses hom instead of $\otimes$. Given a left R-module M, consider the abelian group $\hom_R(S, M)$, where S is regarded as a left R module via restriction. Since the natural right action of S on itself commutes with this left action of R on S, we can make $\hom_R(S, M)$ a left S-module by setting $s * f(-) := f(-.s)$ for any $f \in \hom_R(S, M)$. This S-module is said to be obtained from M by *co-extension of scalars* from R to S.
There is a natural map $\pi\colon \hom_R(S, M) \to M$, given by evaluation at $1 \in S$. Note that

$$r.\pi(f) = r.f(1)$$
$$\pi(r * f) = \pi(\alpha(r) * f) = \pi(f(-.\alpha(r))) = \pi(r.f(-)) = r.f(1)$$

so π is an R-module map if the S-module $\hom_R(S, M)$ is regarded as an R-module by restriction of scalars. Moreover, we have:

Proposition 3.3.4. *Given an S-module N and an R-module map $f\colon N \to M$, there is a unique S-module map $g\colon N \to \hom_R(S, M)$ such that $\pi \circ g = f$.*

$$\begin{tikzcd} & \hom_R(S,M) \arrow[d, "\pi"] \\ N \arrow[ur, dotted, "g"] \arrow[r, "f"'] & M \end{tikzcd}$$

Thus

$$\hom_S(N, \hom_R(S, M)) \approx \hom_R(N, M)$$

so that co-extension of scalars is right adjoint to restriction of scalars

Heuristically, Prop 3.3.4 says that $\hom_R(S, M)$ is the smallest S-module which maps to M by an R-module map.

Proof. Note that g must satisfy $s.g(n) = g(s.n)$ for $s \in S$, $n \in N$; evaluating both sides at 1, we find

$$g(n)(s) = g(s.n)(1) = \pi(g(s.n)) = f(s.n)$$

existence and uniqueness of g follow easily. In fact it's enough to define $g\colon n \mapsto (f(-.n))$ □

Taking $M = N$ (regarded as an R-module) and $f = Id_N$, we obtain from Prop 3.3.4 a canonical S-module map

$$N \to \hom_R(S, N)$$

given by $n \mapsto (s \mapsto s.n)$. This map is injective; moreover, as an R-module map it is split injection.

3.4 Induced and Co-Induced Modules

We now apply the construction we've just seen to ring homomorphisms of the form $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$, where $H \subset G$. In this case extension of scalars (resp. co-extension of scalars) is called **induction** (ref **co-induction**) from H to G. We will often write

$$\mathrm{Ind}_H^G M := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

and

$$\mathrm{Coind}_H^G M := \hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

for an H-module M.

Definition 3.4.1 (Induced Module)**.** A module M is said to be an induced module if it is of the form $M = \mathbb{Z}G \otimes A$, where A is an abelian group and G acts by $g \cdot (r \otimes a) = gr \otimes a$

The definition of co-induced is analogue.
Note that a module of the form $\mathrm{Ind}_{\{1\}}^G M$ is precisely an induced module. Similarly a module of the form $\mathrm{Coind}_{\{1\}}^G M$ will be called co-induced.
Since the right translation action of H on G is free, $\mathbb{Z}G$ is a free right $\mathbb{Z}H$-module; as basis we can take any set E of representatives for the left cosets gH. It follows that $\mathbb{Z}G \otimes_{\mathbb{Z}H} M$, as abelian group, admits a decomposition

$$\mathbb{Z}G \otimes_{\mathbb{Z}H} M = \bigoplus_{g \in E} g \otimes M$$

where $g \otimes M = \{g \otimes m : m \in M\}$ and $g \otimes M \approx M$ via $g \otimes m \mapsto m$. In particular, since we can take 1 as the representative of its coset, it follows that the canonical H-map $i \colon M \to \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ defined in Prop. 3.3.3 , maps M isomorphically onto its image $1 \otimes M$. We can therefore use i to regard M as an H-submodule of $\mathrm{Ind}_H^G M$. Moreover, the summand $g \otimes M$ which occurs above is simply the transform of this submodule under thr action of g, since we set $g \cdot (1 \otimes m) = g \otimes m$. We have therefore established:

Proposition 3.4.1. *The G-module $Ind_H^G M$ contains M as an H-submodule and is the direct sum of the transforms gM, where g ranges over any set of representatives for the left cosets of H in G. In other words, we have*

$$Ind_H^G M = \bigoplus_{g \in G/H} gM$$

This makes sense because M is mapped onto itself by the action of H, so that the subgroup gM of $\mathrm{Ind}_H^G M$ depends only on the class of $g \in G/H$.
The description just given, completely characterizes G-modules of the form $\mathrm{Ind}_H^G M$. More precisely, suppose N is a G-module whose underlying abelian group is a direct sum $\bigoplus_{i \in I} M_i$. Assume that the G-action transitively permutes the summands, in the sense that there is a transitive action of G on I such that $gM_i = M_{gi}$ for all $g \in G$ and $i \in I$. Then we have:

Proposition 3.4.2. *Let N be a G-module as above, let M be one of the summands M_i, and let $H \subset G$ be the isotropy group of i. Then M is an H-module and $N \approx Ind_H^G M$.*

Proof. It is obvious that M is an H-submodule of N, and Prop. 3.3.3 implies that the inclusion $M \hookrightarrow N$ extends to a G-map $\mathrm{Ind}_H^G M \to N$. Clearly φ maps the summand gM of $\mathrm{Ind}_H^G M$ isomorphically onto the corresponding summand M_{gi} of N, so φ is an isomorphism. □

Corollary 3.4.1. *Let N be a G-module whose underlying abelian group is of the form $\bigoplus_{i \in I} M_i$. Assume that the G-action permutes the summands according to some action of G on I. Let G_i be the isotropy group of i and let E be a set of representatives for $I mod G$. Then M_i is a G_i-module and there is a G-isomorphism $N \approx \bigoplus_{i \in E} Ind_{G_i}^G M_i$*

Proof. We have $I = \coprod_{i \in E} Gi$, so $N = \bigoplus_{i \in E} \bigoplus_{j \in Gi} M_j$; now apply Prop 3.4.2 to the inner sum. □

We give now some examples:

Example 3.4.1. The permutation module $\mathbb{Z}[G/H]$ is isomorphic to $\mathrm{Ind}_H^G \mathbb{Z}$, with H acting trivially on $\mathbb{Z}$. This can be seen directly from the definition of $\mathrm{Ind}_H^G \mathbb{Z}$ or, alternatively, by writing $\mathbb{Z}[G/H]$ as a direct sum of copies of $\mathbb{Z}$ and applying Prop 3.4.2.

Example 3.4.2. Let X be a G-CW-complex and consider the G-module $C_n(X)$. This is a direct sum of copies of $\mathbb{Z}$, one for each n-cell of X, and the summands are permuted by the G-action. Hence Prop 3.4.1 gives

$$C_n(X) \approx \bigoplus_{\sigma \in \Sigma_n} \mathrm{Ind}_{G_\sigma}^G \mathbb{Z}_\sigma$$

where Σ_n is a set of representatives for the G-orbits of n-cells,

$$G_\sigma = \{g \in G : g\sigma = \sigma\}$$

and $\mathbb{Z}$ is the *orientation module* associated to σ. i.e. $\mathbb{Z}_\sigma$ is an infinite cyclic group whose two generators corresponds to the two orientation of σ. (Thus $g \in G_\sigma$ acts on $\mathbb{Z}_\sigma$ as $+1$ if g preserves the orientation of σ and -1 otherwise). Note that if G acts freely on X, then the isomorphism above simply reduces to our observation in the section about the Topological Interpretation, that $C_n(X)$ is a free $\mathbb{Z}G$-module with one basis element for each $\sigma \in \Sigma_n$.

For any G-module N we denote by $\mathrm{Res}_H^G N$ the H-module obtained by restriction of scalars from G to H.

3.5 H_* and H^* as Functors of the Coefficient Module

Since $F \otimes_G -$ and $\mathscr{H}_G(F, -)$ are covariant functors, it is clear that $H_*(G, -)$ and $H^*(G, -)$ are covariant functors of the coefficient module. The following proposition gives the basic properties of these functors.

Proposition 3.5.1. *1. There is a natural isomorphism $H_0(G, M) \approx M_G$*

2. There is a natural isomorphism $H^0(G, M) \approx M^G$

3. For any exact sequence $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ of G-modules and any integer n there is a natural map $\partial \colon H_n(G, M'') \to H_{n-1}(G, M')$ such that the sequence

$$\cdots \to H_1(G, M) \to H_1(G, M'') \xrightarrow{\partial} H_0(G, M') \to H_0(G, M) \to H_0(G, M'') \to 0$$

is exact. (The unlabelled arrows here represent the maps induced by i and j.)

4. For any exact sequence $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ of G-modules and any integer n there is a natural map $\delta \colon H^n(G, M'') \to H^{n+1}(G, M')$ such that the sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \xrightarrow{\delta} H^1(G, M') \to H^1(G, M) \to \cdots$$

is exact.

5. *If P is a projective $\mathbb{Z}G$-module then $H_n(G,P)=0$ for $n>0$*

6. *If Q is an injective $\mathbb{Z}G$-module then $H^n(G,Q)=0$ for $n>0$*

Note: the naturality assertion in (3) means that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

with exact rows, the square

$$\begin{array}{ccc} H_n(G,M'') & \xrightarrow{\partial} & H_{n-1}(G,M') \\ \downarrow & & \downarrow \\ H_n(G,N'') & \xrightarrow{\partial} & H_{n-1}(G,N') \end{array}$$

is commutative

Proof. we have already proved (1) and (2). Given an exact sequence as in (3) and a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, we have an exact sequence of chain complexes

$$0 \to M' \otimes_G F \to M \otimes_G F \to M'' \otimes_G F \to 0$$

since projectives are flat; the corresponding long exact homology sequence yields (3). Similarly, (4) follows form the sequence of cochain complexes

$$0 \to \mathscr{H}_G(F,M') \to \mathscr{H}_G(F,M) \to \mathscr{H}_G(F,M'') \to 0$$

which is exact by the definition of projective. Finally, (5), (6) are immediate consequences of the definitions of $H_*(g,-)$ and $H? * (G,-)$ and the exactness of $- \otimes_G P$ and $\hom_G(-,Q)$. $\square$

A functor T (say from R-modules to abelian groups) is said to be effaceable if every module M is a quotient of a module $\bar{M}$ such that $T(\bar{M})=0$. Clearly T is effaceable if $T(P)=0$ for every projective P. Conversely, if T is effaceable then $T(P)=0$ for every projective P. For let $\pi\colon \bar{P} \to P$ be a surjection with $T(\bar{P})=0$; since P is projective, π must split, hence $T(\pi)\colon T(\bar{P}) \to T(P)$ is a split surjection and $T(P)=0$. Thus the content of (5) above is that $H_n(G,-)$ is effaceable for $n>0$. Similarly (6) says that $H^n(G,-)$ is co-effaceable for $n>0$, i.e. that every module M can be embedded in a module $\tilde{M}$ such that $H^n(G,\tilde{M})=0$.

Proposition 3.5.2 (Shapiro's lemma)**.** *If $H \subseteq G$ and M is an H-module, then*

$$H_*(H,M) \approx H_*(G, \mathit{Ind}_H^G M)$$

and

$$H^*(H,M) \approx H^*(G, \mathit{Coind}_H^G M)$$

Proof. Let F be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Then F can also be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$, so

$$H_*(H,M) \approx H_*(F \otimes_{\mathbb{Z}H} M)$$

But $F \otimes_{\mathbb{Z}H} M \approx F \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \approx F \otimes_G (\mathrm{Ind}_H^G M)$, whence the first isomorphism. The second isomorphism follows from the universal property of co-induction, which implies $\mathscr{H}_H(F,M) \approx \mathscr{H}_G(F, \mathrm{Coind}_H^G M)$ (see Prop. 3.3.4) $\square$

Taking $M = \mathbb{Z}$, for example, we conclude from Prop. 3.5.2 that

$$H_*(H) \approx H_*(G, \mathbb{Z}[G/H])$$

(this shows that homology with coefficients is of interests even if one is primarily interested in ordinary integral homology).
Finally, we remark that Shapiro's lemma can be applied with $H = \{1\}$ to yield:

Corollary 3.5.1. *Induced modules $\mathbb{Z}G \otimes A$ are H_*-acyclic. Co-induced modules* $\hom(\mathbb{Z}G, A)$ *are H^*-acyclic.*

4 Spectral Sequences

[Partly taken from Weibel's *An Introduction to Homological Algebra*] In order to motivate the construction of Spectral Sequences, consider the problem of computing the homology of the total chain complex T_* of a first quadrant double complex E_{**}. Recall that each square is anticommutative and that the differential of T_* is the sum of differentials of E_{**}.
As a first step, it is convenient to forget the horizontal differentials (we can name them d^h) and add a superscript zero, retaining only the vertical differential (d^v) along the columns E^0_{p*}.
If we write E^1_{pq} for the vertical homology $H_q(E^0_{p*})$ at the (p, q) spot, we may once again arrange the data in a lattice, this time using horizontal differentials d^h. Now we write E^2_{pq} for the horizontal homology $H_p(E^1_{*q})$ at the (p, q) spot.
In the case of a double complex consisting only of the 2 columns $p, p-1$, it's easy to see that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{pq}$ s.t. $d^v(a) = d^h(a) + d^v(b) = 0$. Unwinding the definition of the differential of the total complex in this particular case, we recognize that $T_* = \mathrm{Cone}(\{d^h\})$ and so we have the l.e.s.

$$\cdots \to H_n(E_{p,*}) \xrightarrow{d^h} H_n(E_{p-1,*}) \to H_n(\mathrm{Cone}(\{d^h\})) \to H_{n-1}(E_{p,*}) \to \cdots$$

which gives us s.e.s

$$0 \to \mathrm{Coker}(d^h_n) \to H_n(\mathrm{Cone}(\{d^h\})) \to \ker(d^h_n) \to 0$$

which, by definition is

$$0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E^2_{pq} \to 0$$

Now we return to the general case of a first quadrant double complex. By diagram chasing, its' easy (and boring) to see that we have an exact sequence

$$H_2(T) \to E^2_{20} \xrightarrow{d} E^2_{0,1} \to H_1(T) \to E^2_{10} \to 0$$

where d is the defined as follows: It can be shown that E^2_{pq} can be presented as the groups of all pairs (a, b) in $E_{p-1,q+1} \times E_{pq}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial $(a, 0); (d^h x, d^v x); (0, d^h c)$ with $x \in E_{p,q+1}$ and $d^v c = 0$. In fact the main idea is the following: take $\bar{t} + B^2_{pq} \in E^2_{pq}$. Then $\bar{t} \in Z^2_{pq}$, so $\tilde{d}^h(\bar{t}) = 0 \in E^1_{p-1,q}$. Moreover, we can represent $\bar{t}$ by $t \in Z^1_{pq}$. Therefore $d^v(t) = 0$. The condition $\tilde{d}^h(\bar{t}) = 0$ means $d^h(t) \in B^1_{p-1,q}$; say $d^h(t) = d^v(s)$ for some $s \in E^0_{p-1,q+1}$. So the pair we are looking for is (s, t). One checks that everything is well-defined and that the association is an isomorphism.
So the map d is simply defined by the formula $d\colon E^2_{p,q} \to E^2_{p-2,q+1}$, $d(a, b) = (0, d^h(a))$.

4.1 Terminology

Definition 4.1.1. A homology spectral sequence (starting with E^a) in an abelian category $\mathcal{A}$ consists of the following data:

1. A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for all integers p, q and $r \geq a$
2. Maps $d^r_{pq} \colon E^r_{pq} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$
3. Isomorphism between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq}:

$$E^{r+1}_{pq} \simeq \ker(d^r_{pq})/\mathrm{Im}(d^r_{p+r,q-r+1})$$

There is a category of homological spectral sequences; a morphism $f \colon A \to E$ is a collection of maps $f^r_{pq} \colon A^r_{pq} \to E^r_{pq}$ in $\mathcal{A}$ with $d^r_E f^r = f^r d^r_A$ such that each f^{r+1}_{pq} is the map induced by f^r_{pq} on homology.

Example 4.1.1. A *first quadrant spectral sequence* is one with $E^r_{pq} = 0$ unless $p, q \geq 0$, that is, the point (p, q) lies in the first quadrant of the plane. (If this condition holds for $r = a$, it clearly holds for all r). If we fix p and q, then $E^r_{pq} = E^{r+1}_{pq}$ for all large r (more specifically $r > \max\{p, q+1\}$), because the d^r landing in (p, q) spot come from the fourth quadrant (i.e is 0), and the d^r leaving E^r_{pq} land in the second quadrant (i.e. is 0). We write E^∞_{pq} for this stable value of E^r_{pq}.

Lemma 4.1.1 (Mapping Lemma)**.** *Let $f \colon \{A^r_{pq}\} \to \{E^r_{pq}\}$ be a morphism of spectral sequences such that for some fixed r, $f^r \colon A^r_{pq} \to E^r_{pq}$ is an isomorphism for all p and q. Then $f^s \colon A^s_{pq} \to E^s_{pq}$ is an isomorphism for all $s \geq r$ as well.*

Proof. By hypothesis we have that f^r_{pq} is an isomorphism for all p, q. Therefore the map induced in homology is an isomorphism and by definition such map is f^{r+1}_{pq}. By induction we have the result. □

Definition 4.1.2 (Bounded Spectral Sequence)**.** A homology spectral sequence is said to be bounded if for each n there are only finitely many nonzero terms of total degree n in E^a_{**}. If so, then for each p and q there is an r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$. We write E^∞_{pq} for this stable value of E^r_{pq}

Definition 4.1.3 (Bounded Convergence)**.** We say that a bounded spectral sequence converges to H_* if we are given a family of objects H_n of $\mathcal{A}$, each having a *finite* filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

and we are given isomorphisms $E^\infty_{pq} \simeq F_p H_{p+q} / F_{p-1} H_{p+q}$. The traditional symbolic way of describing such a bounded convergence is like this

$$E^a_{pq} \Rightarrow H_{p+q}$$

Recall that we can dualize Definition 4.1.1 and obtain a so called *Cohomology Spectral Sequence.* A Cohomology Spectral Sequence is called *bounded* if there are only finitely many nonzero terms in each total degree in E^{**}_a. In a bounded cohomology spectral sequence, we write E^{pq}_∞ for the stable value of the terms E^{pq}_r and say the (bounded) spectral sequence converges to H^* if there is a finite filtration

$$0 = F^t H^n \subseteq \cdots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq F^{p-1} H^n \subseteq \cdots \subseteq F^s H^n = H^n$$

so that $E^{pq}_\infty \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$

Example 4.1.2. If a first quadrant homological spectral sequence converges to H_*, then each H_n has a finite filtration of length $n+1$:

$$0 = F_{-1}H_n \subseteq F_o H_n \subseteq \cdots \subseteq F_{n-1}H_n \subseteq F_n H_n = H_n$$

given by the fact that the n-th degree is the sum of $n+1$ objects plus the zero object, so we take the degree-wise filtration.
The bottom piece $F_0 H_n = E^{\infty}_{0n}$ (recall the definition of convergence: $E^{\infty}_{pq} \simeq F_p H_{p+q}/F_{p-1}H_{p+q}$) of H_n is located on the y-axis, and the top quotient $H_n/F_{n-1}H_n \simeq E^{\infty}_{n0}$ is located on the x-axis. Note that each arrow landing on the x-axis is zero, and each arrow leaving the y-axis is zero because it's a first quadrant homological spectral sequence. Therefore each E^{∞}_{0n} is a quotient of E^a_{0n}, and each E^{∞}_{n0} is a subobject of E^a_{n0}. The terms E^r_{0n} on the y-axis are called the *fiber* terms, and the terms E^r_{n0} on the x-axis are called the *base* terms. The resulting maps $E^a_{0n} \twoheadrightarrow E^{\infty}_{0n} \subset H_n$ (the inclusion is motivated by the fact that the stable page at the spot $(0,n)$ is the first element of the filtration of H_n) and $H_n \twoheadrightarrow E^{\infty}_{n0} \subset E^a_{n0}$ (the first projection is already motivated in the above reasoning, the inclusion is the result of what we have just said) are known as the *edge homomorphisms* of the spectral sequence for the obvious visual reason.

Definition 4.1.4. A (homology) spectral sequence *collapses* at E^r ($r \geq 2$) if there is exactly one non-zero row or column in the lattice $\{E^r_{pq}\}$. The overwhelming majority of all applications of spectral sequences involve spectral sequences that collapse at E^1 or E^2.

Notice that if a collapsing spectral sequence converges to H_*, we can read the H_n off. H_n is the unique non-zero E^r_{pq} with $p+q=n$. In fact the condition $r \geq 2$ ensure that all differentials are zero, hence $E^r_{pq} = E^{\infty}_{pq}$. Now we assume that only the n-th column is non-zero (other case is analogue). We saw that $E^{\infty}_{n0} = H_n/F_{n-1}H_n$, and for $p \leq n-1$, $0 = E^{\infty}_{p,n-p} = F_p H_n/F_{p-1}H_n$, therefore for $p \leq n-1$, $F_p H_n = F_{p-1}H_n$ and by induction they are all trivial. So $H_n = E^{infty}_{n0}$. Now the same reasoning applied to E^{∞}_{pq} shows that the claim holds, and we are done.
Example 4.1.3. (2 columns) Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $p = 0, 1$. We show now that there are exact sequences

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0$$

Clearly, notice that $E^{\infty}_{pq} = E^2_{pq}$ because all the differentials of the second page are zero. By definition we have $E^2_{1,n-1} \simeq F_1 H_n/F_0 H_n$, and we know that $F_0 H_n = E^2_{0n}$. So we have the following short exact sequence

$$0 \to E^2_{0n} \xrightarrow{incl.} H_n \xrightarrow{proj.} E^2_{1,n-1} \to 0$$

Example 4.1.4. (2 rows) Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. We show now that there is a long exact sequence

$$\cdots \to H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \to \cdots$$

Clearly the stable page is the third page, i.e. $E^{\infty}_{pq} = E^3_{pq}$ because of the *slope* of the differential of the third page. Now by definition of the third page, $E^3_{p0} = \ker(d^2 \colon E^2_{p0} \to E^2_{p-2,1})$ and $E^3_{p,1} = \mathrm{Coker}(d^2 \colon E^2_{p+2,0} \to E^2_{p,1})$. Thus we have exact sequences

$$0 \to E^{\infty}_{p,0} \to E^2_{p,0} \xrightarrow{d} E^2_{p-2,1} \to E^{\infty}_{p-2,1} \to 0$$

Now notice that we have the following s.e.s. for any p

$$0 \to F_{p-1}H_p \to H_p \to E^{\infty}_{p,0} \to 0$$

which, after noticing that $0 = E^3_{p-k,k} = E^{\infty}_{p-k,k} \simeq F_{p-k}H_p/F_{p-k-1}H_p$, become

$$0 \to E^{\infty}_{p-1,1} \to H_p \to E^{\infty}_{p,0} \to 0$$

Now glue in the obvious way the sequences obtained so far to get the claim at the beginning

We provide now an intuitive way to think about the convergence of a first quadrant Spectral Sequence.
Consider the stable page:

$$\begin{array}{cccccc}
4 & E^\infty_{04} & \vdots & & & \\
3 & E^\infty_{03} & E^\infty_{13} & \iddots & & \\
2 & E^\infty_{02} & E^\infty_{12} & E^\infty_{22} & \iddots & \\
1 & E^\infty_{01} & E^\infty_{11} & E^\infty_{21} & E^\infty_{31} & \iddots \\
0 & E^\infty_{00} & E^\infty_{10} & E^\infty_{20} & E^\infty_{30} & \cdots \\
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

We know that $E^\infty_{00} = \frac{F_0H_0}{F_{-1}H_0}$. By the fact that the second quadrant is all zero, in particular $E^\infty_{-n,n} = 0$ for any $n \in \mathbb{N}_{>0}$, we have that (inductively) $F_{-1}H_0 = 0$. Hence $E^\infty_{00} = H_0$ as we saw before. Now we start with the interesting cases, for the very same reasoning as in the case of E^∞_{00}, $E^\infty_{01} = F_0H_1$, by definition $E^\infty_{10} = \frac{F_1H_1}{F_0H_1} = \frac{F_1H_1}{E^\infty_{01}}$. Now notice that (due to the fact the the fourth quadrant is all zero) $F_1H_1 = H_1$, hence we have

$$E^\infty_{10} = \frac{H_1}{E^\infty_{01}}$$

which is equivalent to say that we have a s.e.s

$$0 \to E^\infty_{01} \to H_1 \to E^\infty_{10} \to 0$$

Hence H_1 has to be an extension of E^∞_{10} with respect to E^∞_{01}. Now we will deal with the next diagonal, which will show how the general case works: as before $E^\infty_{02} = F_0H_2$, and $E^\infty_{20} = \frac{F_2H_2}{F_1H_2} = \frac{H_2}{F_1H_2}$, notice that by definition $E^\infty_{11} = \frac{F_1H_2}{F_0H_2}$. So we can fit all of these informations together in the following s.e.s

$$\begin{gathered}
0 \to E^\infty_{02} \to F_1H_2 \to E^\infty_{11} \to 0 \\
0 \to F_1H_2 \to H_2 \to E^\infty_{20} \to 0
\end{gathered}$$

Which shows what's the general way to think about of (first quadrant) Spectral Sequences: once you reach the stable page, just solve *consecutive* extension problems starting from E^∞_{0p}, and the last extension will provide you the right result. Notice that this is a theoretical method only, it's very difficult in practice knowing what's the right extension to choose at every step, and every wrong decision one makes, will inevitably screw all computations. Still, this method has is value, when for example, we have additional hypothesis on the s.e.s., if we know that they all split for example (if we are working in $\mathbf{Vect}_K$ for example) then by the above reasoning

$$H_k \cong \bigoplus_{p+q=k, p,q \geq 0} E^\infty_{pq}$$

Given a homology spectral sequence, we see that each E^{r+1}_{pq} is a subquotient of the previous term E^r_{pq}. By induction on r, we see that there is a nested family of subobjects of E^a_{pq}:

$$0 = B^a_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq B^{r+1}_{pq} \subseteq \cdots \subseteq Z^{r+1}_{pq} \subseteq Z^r_{pq} \subseteq \cdots \subseteq Z^a_{pq} = E^a_{pq}$$

such that $E^r_{pq} \cong Z^r_{pq}/B^r_{pq}$. The inductive step is done as follows:

$$
\begin{tikzcd}
E^r_{p_1q_1} \arrow[rr, "d_{p_1q_1}"] \arrow[dr, twoheadrightarrow, "d_{p_1q_1}"'] & & E^r_{pq} \arrow[rr, "d_{pq}"] & & E^r_{p_2q_2} & \\
& \tilde{B}^{r+1}_{pq} \arrow[ur, hook] \arrow[rr, hook] & & \tilde{Z}^{r+1}_{pq} \arrow[ul, hook'] \arrow[rr, twoheadrightarrow, "\pi"] & & E^{r+1}_{pq} = \mathrm{Coker}(\pi)
\end{tikzcd}
$$

We introduce the intermediate objects

$$
B^\infty_{pq} = \bigcup_{r=a}^{\infty} B^r_{pq} \qquad \text{and} \qquad Z^\infty_{pq} = \bigcap_{r=a}^{\infty} Z^r_{pq}
$$

and define $E^\infty_{pq} = Z^\infty_{pq}/B^\infty_{pq}$. In a bounded spectral sequence both the union and the intersection are finite, so $B^\infty_{pq} = B^r_{pq}$ and $Z^\infty_{pq} = Z^r_{pq}$ for large r. Thus we recover our earlier definition of E^∞_{pq}. Notice that we are tacitly assume that Z^∞_{pq}, B^∞_{pq} exist. In $\mathbf{Mod}_R$ this is always true, for a general abelian category one has to assume some additional hypothesis.

4.2 The Leray-Serre Spectral Sequence

Definition 4.2.1 (Serre Fibration)**.** A sequence

$$
(F, *_F) \xrightarrow{i} (E, *_E) \xrightarrow{\pi} (B, *_B)
$$

of based topological spaces is called a Serre fibration if F is the inverse image of $\pi^{-1}(*_B)$ and if π has the following *Homotopy Lifting Property* (HLP):
If P is any finite polyhedron and I is the unit interval $[0,1]$, $g\colon P \to E$ is a map, and $H\colon P \times I \to B$ is a homotopy between $\pi \circ g = H(-,0)$ and $h_1 = H(-,1)$

$$
\begin{tikzcd}
P \arrow[r, "g"] \arrow[d, "i_0"] & E \arrow[d, "\pi"] \\
P \times I \arrow[r, "H"] \arrow[ur, "G"] & B
\end{tikzcd}
$$

(i_0 is the inclusion at level 0) there is a homotopy $G\colon P \times I \to E$ between g and a map $g_1 = G(-,1)$ which lifts H in the sense that $\pi \circ G = H$.
The spaces F, E and B are called the *Fiber, total space* and *Base space* respectively.

The importance of Serre fibrations lies in the fact (proven in Serre's thesis) that associated to each fibration is a long exact sequence of homotopy groups

$$
\cdots \to \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial} \cdots
$$

In order to simplify the presentation below, we shall assume that B is simply connected. Without this assumption, we would have to introduce the action of $\pi(B)$ on the homology of F and talk about the homology of B with *local coefficients* in the twisted bundles $H_q(F)$.

Theorem 4.2.1 (Leray-Serre spectral sequence)**.** *Let $(F, *_F) \xrightarrow{i} (E, *_E) \xrightarrow{\pi} (B, *_B)$ be a Serre Fibration such that B is simply connected. Then there is a first quadrant homology spectral sequence starting with E^2 and converging to $H_*(E)$:*

$$
E^2_{pq} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)
$$

$$
\begin{array}{cccccc}
4 & E^2_{04} & \vdots & & & \\
3 & E^2_{03} & E^2_{13} & \iddots & & \\
2 & E^2_{02} & E^2_{12} & E^2_{22} & \iddots & \\
1 & E^2_{01} & E^2_{11} & E^2_{21} & E^2_{31} & \iddots \\
0 & E^2_{00} & E^2_{10} & E^2_{20} & E^2_{30} & \cdots \\
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Remark 4.2.1. $H_0(B) = \mathbb{Z}$, so along the y-axis we have $E^2_{0q} = H_q(F)$. Because $E^2_{pq} = 0$ for $p < 0$, the groups $E^3_{0q}, \ldots, E^{n+1}_{0q} = E^\infty_{0q}$ are successive quotients of E^2_{0q}. The theorem states that $E^\infty_{0q} \simeq F_0 H_q(E)$, so there is an edge map

$$H_q(F) = E^2_{0q} \twoheadrightarrow E^\infty_{0q} \subseteq H_q(E)$$

This edge map is the map $i_* \colon H_q(F) \to H_q(E)$

Remark 4.2.2. Suppose that $\pi_0(F) = 0$, so that $H_0(F) = \mathbb{Z}$. Along the x-axis we then have $E^2_{p0} = H_p(B)$. Because $E^2_{pq} = 0$ for $q < 0$, the groups $E^3_{p0}, \ldots, E^{n+1}_{p0} = E^\infty_{p0}$ are successive subgroups of E^2_{p0}. The theorem states that $E^\infty_{p0} \simeq H_p(E)/F_{p-1}H_p(E)$, so there is an *edge map*

$$H_p(E) \twoheadrightarrow E^\infty_{p0} \hookrightarrow E^2_{p0} = H_p(B)$$

This edge map is the map $\pi_* \colon H_p(E) \to H_p(B)$.

Remark 4.2.3. The Universal Coefficient Theorem tells us that

$$H_p(B; H_q(F)) \simeq H_p(B) \otimes H_q(F) \;\oplus\; \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(B), H_q(F))$$

Therefore the terms E^2_{pq} are not hard to calculate. In particular, since $\pi_1(B) = 0$ we have $H_1(B) = H_1(B; H_q(F)) = 0$ for all q. By the Hurewicz homomorphism, $\pi_2(B) \simeq H_2(B)$ and therefore $H_2(B; H_q(F)) \simeq H_2(B) \otimes H_q(F)$ for all q as well.

Example 4.2.1 (Loop Spaces). Let PB denote the space of based paths in B, that is, maps $[0,1] \to B$ sending 0 to $*_B$ with the compact-open topology. The subspace of based loops in B (maps $[0,1] \to B$ sending 0 and 1 to $*_B$) is written ΩB. There is a fibration

$$\Omega B \hookrightarrow PB \xrightarrow{\mathrm{ev}_1} B$$

where ev_1 is evaluation at 1. The space PB is contractible, because paths may be pulled back along themselves to the basepoint, so $H_n(PB) = 0$ for $n \neq 0$. Therefore, except for $E^\infty_{00} = \mathbb{Z}$, we have a spectral sequence converging to zero.

$$
\begin{tikzcd}
4 & \vdots & \vdots & & & \\
3 & H_3(\Omega B) & 0 & & & \\
2 & H_2(\Omega B) & 0 & \vdots & & \\
1 & H_1(\Omega B) & 0 & E^2_{21} \arrow[ull] & & \\
0 & \mathbb{Z} & 0 & H_2(B) \arrow[ull] & H_3(B) \arrow[ull] & H_4(B) \arrow[ull] \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

From the low degree terms (assuming $\pi_1(B) = 0!$), we see that $H_1(\Omega B) \simeq H_2(B)$: Call $d_1^2 \colon H_2(B) \to H_1(\Omega B)$, because the differential from $H_1(\Omega B)$ lands in a zero, and the differential to $H_2(B)$ start from a zero, we have that $E^3_{01} \simeq \operatorname{Coker} d_1^2$ and $E^3_{20} \simeq \ker d_1^2$. Now being $E^\infty_{pq} = 0$ for any pair of indices (except in 00) we have that $\ker d^2$ and $\operatorname{Coker} d_1^2$ are trivial, hence d_1^2 is an isomorphism.
By UCT and the hypothesis on $\pi_1(B)$, we have that $H_2(B, H_2(B)) \simeq H_2(B) \otimes H_2(B)$.
We claim that

$$H_4(B) \xrightarrow{d_2^2} H_2(B) \otimes H_2(B) \xrightarrow{d_3^2} H_2(\Omega B) \to H_3(B) \to 0$$

is exact. Exactness in $H_2(B, H_2(B)) \simeq H_2(B) \otimes H_2(B)$ is a consequence of the fact that $E^3_{21} = E^\infty_{21} = 0$. Now to conclude it's enough to notice that $H_3(B) \simeq \operatorname{Coker} d_3^2$. Consider the third page of the spectral sequence:

$$\begin{tikzcd}
2 & \operatorname{Coker} d_3^2 & 0 & & & \\
1 & E^3_{01} & 0 & & & \\
0 & \mathbb{Z} & 0 & & H_3(B) \arrow[uulll] & \\
& 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

We have the sequence $0 \to H_3(B) \xrightarrow{d_1^3} \operatorname{Coker} d_2^2 \to 0$. With a analogue reasoning as above, and noticing that both E^4_{30} and E^4_{02} are stable value and therefore trivial we have exactness as claimed, and therefore $H_3(B) \simeq \operatorname{Coker} d_2^2$.
As a final note we can use induction on n to estimate the size of $H_n(\Omega B)$.

Example 4.2.2. We apply what we have just seen to compute the homology of ΩS^n, for $n \geq 2$. We will depict here the case $n = 3$ but nevertheless the machinery we will develop here will prove the claim for any $n \geq 2$. So consider the second page of the Leray-Serre Spectral Sequence for $\Omega S^3 \to PS^3 \to S^3$

$$\begin{tikzcd}
4 & H_4(\Omega S^4) & 0 & 0 & H_4(\Omega S^3) & 0 \\
3 & H_3(\Omega S^4) & 0 & 0 & H_3(\Omega S^3) & 0 \\
2 & H_2(\Omega S^4) & 0 & 0 & H_2(\Omega S^3) & 0 \\
1 & H_1(\Omega S^4) & 0 & 0 & H_1(\Omega S^3) & 0 \\
0 & \mathbb{Z} & 0 & 0 \arrow[ull] & \mathbb{Z} & 0 \\
& 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

Being E^3_{01} and E^3_{10} stable elements (if we go further with the pages all differentials from there and to there are zero), we automatically have that they are trivial and therefore the arrow is an iso.
With the same argument, looking at the n-th page, (here $n = 3$) we have

$$\begin{tikzcd}
4 & H_4(\Omega S^4) & 0 & 0 & H_4(\Omega S^3) & 0 \\
3 & H_3(\Omega S^4) & 0 & 0 & H_3(\Omega S^3) & 0 \\
2 & H_2(\Omega S^4) & 0 & 0 & H_2(\Omega S^3) \arrow[uulll] & 0 \\
1 & 0 & 0 & 0 & H_1(\Omega S^3) \arrow[uulll] & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \arrow[uulll] & 0 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

With the same argument as above, we see that the $n+1$-th page is stable (and all zero expect in the origin), therefore all the arrows here are isomorphism. Hence we have computed the homology of the loop space ΩS^n which is

$$H_p(\Omega S^n) \simeq \begin{cases} \mathbb{Z} & \text{if } (n-1) \text{ divides } p \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.2.3 (Wang Sequence). If $F \xrightarrow{i} E \xrightarrow{\pi} S^n$ is a fibration whose base space is an n-sphere $(n \neq 0, 1)$, there is a long exact sequence

$$\cdots \longrightarrow H_q(F) \xrightarrow{i} H_q(E) \longrightarrow H_{q-n}(F) \xrightarrow{d^n} H_{q-1}(F) \xrightarrow{i} H_{q-1}(E) \longrightarrow \cdots$$

In particular, $H_q(F) \simeq H_q(E)$ if $0 \leq q \leq n-2$. As before we depict the situation in the second page for $n = 3$:

$$\begin{array}{cccccc}
4 & H_4(F) & \vdots & \vdots & H_4(F) & \vdots \\
3 & H_3(F) & 0 & 0 & H_3(F) & 0 \\
2 & H_2(F) & 0 & 0 & H_2(F) & 0 \\
1 & H_1(F) & 0 & 0 & H_1(F) & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 \\
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

notice that the third page is the same, because all differential are zero before such page. In general, for any n, we can directly go to the n-th page.

$$\begin{tikzcd}
 & & & & \\
3 & H_3(F) & 0 & 0 & H_3(F) \\
2 & H_2(F) & 0 & 0 & H_2(F) \arrow[uulll] \\
1 & H_1(F) & 0 & 0 & H_1(F) \arrow[uulll] \\
0 & H_0(F) & 0 & 0 & H_0(F) \arrow[uulll] \\
 & 0 & 1 & 2 & 3
\end{tikzcd}$$

We clearly have, for all q, the exact sequences

$$0 \longrightarrow \ker d^3 \hookrightarrow H_q(F) \xrightarrow{d^3} H_{q+2}(F) \twoheadrightarrow \operatorname{Coker} d^3 \longrightarrow 0$$

which for general n, they become

$$0 \longrightarrow \ker d^n \hookrightarrow H_q(F) \xrightarrow{d^n} H_{q+n-1}(F) \twoheadrightarrow \operatorname{Coker} d^n \longrightarrow 0$$

Now looking at differentials in the page $n+1$ ad so on, it's clear that they are all zero. Let $d_n\colon E^n_{n,q-n} \to E^n_{0,q+n-1}$, by the reasoning just said $\ker d^n \simeq E^\infty_{n,q-n}$ (we decide to call the second index $q-n$ in order to simplify some notation) and $\operatorname{Coker} d^n \simeq E^\infty_{0,q+n-1}$. Now recalling the definition of convergence, we have

$$E^\infty_{n,q-n} \simeq \frac{F_n H_q(E)}{F_{n-1}H_q(E)}$$

using that $E^\infty_{n+i,q-n-i} = 0$, we can conclude that $F_n H_q(E) \simeq H_q(E)$. Using instead $0 = E^\infty_{j,q-j} \simeq \frac{F_j H_q(E)}{F_{j-1}H_q(E)}$ for $1 \le j \le n-1$ we can conclude that $F_{n-1}H_q(E) = F_0 H_q(E)$. So we have s.e.s

$$0 \to E^\infty_{0q} \to H_q(E) \to E^\infty_{n,q-n} \to 0$$

The Wang sequence is now obtained by splicing together these two families of short exact sequences.

Example 4.2.4 (Gysin Sequence). If $S^n \to E \xrightarrow{\pi} B$ is a fibration with B simply connected and $n \neq 0$, there is an exact sequence

$$\cdots \longrightarrow H_{p-n}(B) \longrightarrow H_p(E) \xrightarrow{\pi} H_p(B) \xrightarrow{d^{n+1}} H_{p-n-1}(B) \longrightarrow H_{p-1}(E) \xrightarrow{\pi} \cdots$$

In particular $H_p(E) \simeq H_p(B)$ for $0 \le p < n$.
The proof is similar to the Wang sequence, except that now the nonzero terms E^2_{pq} all lie on the two rows $q = 0, n$. The only nontrivial differentials are d^{n+1}_{p0} from $H_p(B) = E^{n+1}_{p0}$ to $E^{n+1}_{p-n-1,n} \simeq H_{p-n-1}(B)$.

4.3 The Leray-Hochschild-Serre Sequence

Lemma 4.3.1. *If H is a normal subgroup of G, and A is a G-module, then both A_H and A^H are G/H-modules. Moreover, the forgetful functor $\rho^\sharp$ from G/H-modules to G-modules has $(-)_H$ as left adjoint and $(-)^H$ as right adjoint.*

Proof. A G/H-module is the same thing as a G-module on which H acts trivially. Therefore A_H and A^H are G/H-modules by construction. The universal properties of $A^H \to A$ and $A \to A_H$ translate into natural isomorphisms

$$\hom_G(A, \rho^\sharp B) \simeq \hom_{G/H}(A_H, B)$$
$$\hom_G(\rho^\sharp B, A) \simeq \hom_{G/H}(B, A^H)$$

We show now the first equivalence, the second is similar. Take $f\colon A \to \rho^\sharp B$. We have $f(h.a) = h.f(a) = f(a)$, hence f can be seen as a G/H map from $A_H \to B$ (because $f(h.a) = f(a)$ and f is a G-map). In the other direction, take $f\colon A_H \to B$, we need to prove that it can be related to a G-map between A and $\rho^\sharp$. Notice that $f(g.a) = f(gH.a) = gH.f(a) = g.f(a)$, and hence we are done.
Now naturality means prove the commutativity of the following diagram. Let $\varphi\colon A \to A'$ and $\psi\colon B \to B'$, we have:

$$\begin{array}{ccccc}
\hom_{G/H}(A_H, B) & \xrightarrow{-\circ\varphi_H} & \hom_{G/H}(A'_H, B) & \xrightarrow{\psi\circ -} & \hom_{G/H}(A'_H, B') \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
\hom_G(A, \rho^\sharp B) & \xrightarrow[-\circ\varphi]{} & \hom_G(A', \rho^\sharp B) & \xrightarrow[\rho^\sharp\psi\circ -]{} & \hom_G(A', \rho^\sharp B')
\end{array}$$

which is clearly commutative. □

Theorem 4.3.1 (Lyndon-Hochschild-Serre Spectral Sequence)**.** *For every normal subgroup H of a group G there are two convergent first quadrant spectral sequences:*

$$E^2_{pq} = H_p(G/H; H_q(H; A)) \Rightarrow H_{p+q}(G; A)$$
$$E_2^{pq} = H^p(G/H; H^q(H; A)) \Rightarrow H^{p+q}(G; A)$$

Proof. (Homology) We want to retrieve it as a special case of the well know Grothendieck Spectral Sequence for the following commutative diagram (commutativity is obvious).

$$\begin{tikzcd}
G-\mathbf{mod} \arrow[rr, "(-)_H"] \arrow[dr, "(-)_G"'] & & G/H-\mathbf{mod} \arrow[dl, "(-)_{G/H}"] \\
& \mathbf{Ab} &
\end{tikzcd}$$

The adjunction shown in the lemma, take care of the fact that in order to apply Grothendieck, the functors have to be right exact and $(-)_H$ has to map projective in G-**mod** to $(-)_{G/H}$-acyclic objects in G/H-**mod** (in fact we proved more: it preserves projective).
(Cohomology) As before we want to use the Grothendieck Spectral Sequence for the following commutative diagram.

$$\begin{tikzcd}
G-\mathbf{mod} \arrow[rr, "(-)^H"] \arrow[dr, "(-)^G"'] & & G/H-\mathbf{mod} \arrow[dl, "(-)^{G/H}"] \\
& \mathbf{Ab} &
\end{tikzcd}$$

The adjunction shown in the lemma, take care of the fact that in order to apply Grothendieck, the functors have to be left exact and $(-)^H$ has to map injectives in G-**mod** to $(-)^{G/H}$-acyclic objects in G/H-**mod** (in fact we proved more: it preserves injectives). □

5 Finiteness Condition

5.1 Introduction

Recall that the definition of $H_*(G, M)$ and $H^*(G, M)$ allows us to choose an arbitrary projective resolution $P = (P_i)_{i\geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Similarly, if we wish to take the topological point of view, then we can compute $H_*(G, M)$ and $H^*(G, M)$ in terms of an arbitrary $K(G, 1)$-complex Y. Since we have this freedom of choice, it is reasonable to try to choose P (or Y) to be as *small* as possible, and this leads to various finiteness condition on G.
For example, if we interpret *small* in terms of the length of P (or the dimension of Y), then we are led to the notion of *cohomological dimension*. Or if we interpret small t mean that each P_i

should be finitely generated (or that Y should have only finitely many cells), then we are led to the so-called FP and FL conditions.
Our goal in this section is to introduce these and related finiteness conditions and to give some examples. Our treatment will by no means be complete. Finally a word about notation: In the theory of discrete subgroups of Lie groups, which is the source of our examples, it is customary to denote the Lie group by G and the discrete subgroup by Γ. In order to be consistent with this, we will use the letter Γ from now on (instead of G) to denote a typical abstract group.

5.2 Cohomological Dimension

5.2.1 Some Basic Stuff

We begin with a basic lemma which characterizes projective dimension in terms of the vanishing of cohomology functors.

Definition 5.2.1. If R is a ring and M a R-module, the projective dimension of M, denote proj dim$M$ is defined to be the infimum of the set of integers n such that M admits a projective resolution $0 \to P_n \to \cdots \to P_0 \to M \to 0$ of length n.

Recall also that the Ext functors are defined by

$$\mathrm{Ext}^i_R(M, -) := H^i(\mathscr{H}_R(P, -))$$

where P is a projective resolution of M. In particular,

$$\mathrm{Ext}^i_{\mathbb{Z}\Gamma}(\mathbb{Z}, -) = H^i(\Gamma, -)$$

We first start with this easy lemma

Lemma 5.2.1. *If M is an R-module, the following conditions are equivalent*

1. *M is projective*
2. *$Ext^n_R(M, N) = 0$ for all $n \geq 1$ and all R-modules N*
3. *$Ext^1_R(M, N) = 0$ for all R-modules N*

Proof. Clearly the only non trivial proof is (3) $\Rightarrow$ (1). Take any short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

and consider the related l.e.s

$$0 \to \hom_R(M, A) \to \hom_R(M, B) \to \hom_R(C,) \to \mathrm{Ext}^1_R(M, A) \to \cdots$$

By hypothesis $\mathrm{Ext}^1_R(M, A) = 0$, hence starting from a s.e.s. as above, we obtain

$$0 \to \hom_R(M, A) \to \hom_R(M, B) \to \hom_R(M, C) \to 0$$

hence $\hom_R(M, -)$ is an exact functor which means that M is projective. □

Lemma 5.2.2. *The following conditions are equivalent:*

1. *proj $dim_R M \leq n$*
2. *$Ext^i_R(M, N) = 0$ for $i > n$ and every R-module N*
3. *$Ext^{n+1}_R(M, -) = 0$ for every R-module N.*

4. *If $0 \to K_{n-1} \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is any exact sequence of R-modules with each P_i projective, then K_{n-1} is projective.*

Proof. It is obvious that $4 \Rightarrow 1 \Rightarrow 2 \Rightarrow 3$, so we need only to prove $3 \Rightarrow 4$. Given a partial resolution as in 4, break it into short exact sequences. We start in degree 0 and we obtain

$$0 \longrightarrow K_0 \xhookrightarrow{i_0} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

Where K_0 is the kernel of ϵ. The induced long exact sequence is

$$\cdots \longrightarrow \mathrm{Ext}^n_R(P_0, N) \longrightarrow \mathrm{Ext}^n_R(K_0, N) \longrightarrow \mathrm{Ext}^{n+1}_R(M, N) \longrightarrow \mathrm{Ext}^{n+1}_R(P_0, N) \longrightarrow \cdots$$

Now if every third term in an exact sequence is =, then the maps in the middle are both injective and surjective, hence isomorphisms. This is precisely what we have here, because P_0 is projective, and thus $\mathrm{Ext}^n_R(P_0, N) = 0$ for all $n \geq 1$. Therefore, $\mathrm{Ext}^{n+1}_R(M, N) \simeq \mathrm{Ext}^n_R(K_0, N)$, so as we slide from right to left through the exact sequence, the upper index decreases by 1. This technique is referred to as dimension shifting.
Now the second short exact sequence is

$$0 \longrightarrow K_1 \xhookrightarrow{i_1} P_1 \xrightarrow{d_1} K_0 \longrightarrow 0$$

Where K_1 is the kernel of d_1, and d_1 is surjective because by exactness, $\mathrm{Im}(d_1) = \ker(\epsilon) = K_0$. The associated l.e.s. is

$$\cdots \longrightarrow \mathrm{Ext}^n_R(P_1, N) \longrightarrow \mathrm{Ext}^n_R(K_1, N) \longrightarrow \mathrm{Ext}^{n+1}_R(K_0, N) \longrightarrow \mathrm{Ext}^{n+1}_R(P_1, N) \longrightarrow \cdots$$

and dimension shifting gives $\mathrm{Ext}^n_R(K_0, N) \simeq \mathrm{Ext}^{n-1}_R(K_1, N)$. Iterating this procedure, we get $\mathrm{Ext}^{n+1}_R(M, N) \simeq \mathrm{Ext}^1_R(K_{n-1}, N)$, hence by hypothesis of (3), $\mathrm{Ext}^1_R(K_{n-1}, N) = 0$. So by Lemma 5.2.1 K_{n-1} is projective, and hence we are done. $\square$

The implication $1 \Rightarrow 4$ is very useful. It shows that if there exist a projective resolution of length n, then we don't have to be clever to find one - *any* partial resolution of length $n-1$ can be completed to a projective resolution of length n.
We now specialize to the case $R = \mathbb{Z}\Gamma$, $M = \mathbb{Z}$.

Definition 5.2.2 (Cohomological Dimension)**.** The cohomological dimension of Γ, denoted $\mathrm{cd}\Gamma$, is defined to be the smallest integer n such that the condition of Lemma 5.2.2 hold (for $R = \mathbb{Z}\Gamma$ and $M = \mathbb{Z}$), provided there exist such integer n; otherwise we set $\mathrm{cd}\Gamma = \infty$.

Clear from the definition we have

$$\begin{aligned} \mathrm{cd}\Gamma &= \text{proj } \dim_{\mathbb{Z}\Gamma}\mathbb{Z} \\ &= \inf\{n \mid \mathbb{Z} \text{ adimts a proj. res. of length } n\} \\ &= \inf\{n \mid H^i(\Gamma, -) = 0 \text{ for } i > n\} \\ &= \sup\{n \mid H^(\Gamma, M) \neq 0 \text{ for some } \Gamma - \text{module } M\} \end{aligned}$$

There is an obvious topological analogue of $\mathrm{cd}\Gamma$: the *geometric dimension* of Γ, denoted geom $\dim\Gamma$, is defined to be the minimal dimension of a $K(\Gamma, 1)$-complex. Since the cellular chain complex of the universal cover of a $K(\Gamma, 1)$-complex Y yields a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ (of length equal to the dimension of Y), we clearly have:

Proposition 5.2.1.

$$cd\Gamma \leq geom\ dim\Gamma$$

Example 5.2.1. $\mathrm{cd}\Gamma = 0$ if and only if Γ is the trivial group.
One direction is easy, we assume now that $\mathrm{cd}\Gamma = 0$. This means that $\mathbb{Z}$ is a projective $\mathbb{Z}\Gamma$-module, which means that the canonical surjection splits $\epsilon\colon \mathbb{Z}\Gamma \to \mathbb{Z}$.
So suppose that the augmentation $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$, $g \mapsto 1$ splits. This means there's a map of $\mathbb{Z}\Gamma$-modules $j : \mathbb{Z} \to \mathbb{Z}\Gamma$ such that $\epsilon(j(n)) = n$ for all $n \in \mathbb{Z}$. Let $z = j(1) \in \mathbb{Z}\Gamma$. Then $j(g.1) = j(1) = z = g.j(1) = g \cdot z$ for all g, hence $z \in \mathbb{Z}\Gamma^\Gamma$ is invariant.
This is only possible if Γ is finite, and z has to be a scalar multiple of $\xi = \sum_{g \in \Gamma} g \in \mathbb{Z}\Gamma$ (easy to check), say $z = n\xi$ for some $n \in \mathbb{Z}$. Clearly $\epsilon(z) = n \cdot |\Gamma|$, but at the same time $z = j(1) \implies \epsilon(z) = 1$ (since j is a splitting). Hence $n \cdot |\Gamma| = 1$, which can only happen if $|\Gamma| = 1$.

The rest of the section will be devoted to some elementary properties of cohomological dimension.

Proposition 5.2.2. *If $cd\Gamma < \infty$ then*

$$cd\Gamma = \sup\{n \mid H^n(\Gamma, F) \neq 0 \textit{ for some free } \mathbb{Z}\Gamma - \textit{module } F\}$$

Proof. Let $n = \mathrm{cd}\Gamma$. In view of the long exact cohomology sequence, the functor $H^n(\Gamma, -)$ is right exact. Since $H^n(\Gamma, M) \neq 0$ for some M, if follows that $H^n(\Gamma, F) \neq 0$ for any free module F which maps onto M. ☐

Proposition 5.2.3. *1. If $\Gamma' \subset \Gamma$ then*

$$cd\Gamma' \leq cd\Gamma$$

equality holds if $cd\Gamma < \infty$ and $[\Gamma\colon \Gamma'] < \infty$.

2. *If $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ is a s.e.s. of groups, then*

$$cd\Gamma \leq cd\Gamma' + cd\Gamma''$$

3. *if $\Gamma = \Gamma_1 *_A \Gamma_2$ (where $A \hookrightarrow \Gamma_i$), then*

$$cd\Gamma \leq \max\{cd\Gamma_1, cd\Gamma_2, 1 + cdA\}$$

Proof. The inequality in (1) follows immediately from Shapiro's Lemma 3.5.2 or, alternatively, from the fact that a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ can also be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$. To prove the second part of (1), suppose $\mathrm{cd}\Gamma = n < \infty$. By 5.2.2 there is a free $\mathbb{Z}\Gamma$-module F with $H^n(\Gamma, F) \neq 0$. If F' is a free $\mathbb{Z}\Gamma'$-module of the same rank, then $F \approx \mathrm{Ind}_{\Gamma'}^{\Gamma} F'$, so Shapiro's lemma yields $H^n(\Gamma', F') \simeq H^n(\Gamma, F) \neq 0$. Thus $\mathrm{cd}\Gamma' \geq n$, whence (1).
(2) is an immediate consequence of the Hochschild-Serre spectral sequence. (3) follows from the cohomology version of the Mayer-Vietoris sequence. ☐

Corollary 5.2.1. *If $cd\Gamma < \infty$, then Γ is torsion-free.*

Proof. If Γ is not torsion-free, then Γ contains a nontrivial finite cyclic subgroup Γ'. Such a Γ' has $\mathrm{cd}\Gamma' = \infty$ since $H^{2k}(\Gamma', \mathbb{Z}) \neq 0$ for all k. So 5.2.3 (1) implies that $\mathrm{cd}\Gamma = \infty$. ☐

Our last result shows that, as far as cohomological dimension is concerned, we never need to use projective resolutions which are not free:

Proposition 5.2.4. *For any group Γ there is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of length equal to $cd\Gamma$.*

The proof requires the following *Eilenberg trick*:

Lemma 5.2.3. *If P is a projective module over a arbitrary ring R, then there is a free module F such that $P \oplus F \simeq F$.*

(warning: F will be of infinite rank, in general, even if P is finitely generated).

Proof. Since P is projective there is a module Q such that $P \oplus Q$ is free. Let F be the countable sum

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \cdots$$

Then F is free, being a direct sum of free modules. But F can also be described as the sum of a countable number of copies of P and a countable number of copies of Q. Adding one more copy of P doesn't change this, so $P \oplus F \simeq F$. □

Proof. (Proposition 5.2.4 Let $n = \mathrm{cd}\Gamma$. we may assume $0 < n < \infty$. Choose a partial free resolution

$$F_{n-1} \xrightarrow{\partial} \cdots \to F_0 \to \mathbb{Z} \to 0$$

of length $n-1$ and let $P = \ker\{\partial colon F_{n-1} \to F_{n-2}\}$. (Here we set $F_{-1} = \mathbb{Z}$ if $n = 1$). Then P is projective by 5.2.2, so Lemma 5.2.3 gives us a free module F such that $P \oplus F$ is free. Thus is we replace F_{n-1} by $F_{n-1} \oplus F$ and set $\partial_{|F} = 0$, we obtain a partial free resolution of length $n-1$ with $\ker \partial$ free, whence the proposition □

5.2.2 Serre's Theorem

Theorem 5.2.1. *If Γ is a torsion-free group and Γ' is a subgroup of finite index, then*

$$cd\Gamma' = cd\Gamma$$

Be careful on the hypothesis of being torsion-free. If we drop such hypothesis then the Theorem is highly false, just consider the trivial group as a subgroup of some finite cyclic groups.

Proof. In view of 5.2.3.1, we need to show that if $\mathrm{cd}\Gamma' < \infty$, then $\mathrm{cd}\Gamma < \infty$. (In fact, if we prove this fact, then we can apply the first point of the proposition because the two hypothesis: finiteness of the index and finiteness of the cohomological dimension are satisfied). We will give a topological proof of it. This reasoning will use a result which will be proved later, namely the theorem of Eilenberg and Ganea: If Γ is a group such that $\mathrm{cd}\Gamma < \infty$, then there exists a finite dimensional $K(\Gamma, 1)$-complex.
Returning now to the proof of 5.2.1, we are given that $\mathrm{cd}\Gamma' < \infty$, so (by Eilenberg-Ganea) there is a finite-dimensional $K(\Gamma', 1)$. Its universal cover X' is then a finite-dimensional (by definition of CW-structure induce by projection it has the same dimension -although much more cells on each dimension-), contractible (because all homotopy groups vanish), free Γ'-complex. In fact recall that $G' \simeq \pi_1(K(G', 1))$ acts freely on the cells on any regular coverings, for example on the universal covering for the following easy reason: By the fact that Deck Transformations composed with the projection p are the identity, if $\sigma \in p^{-1}(\alpha)$ then even $g.\sigma \in p^{-1}(\alpha)$. Now assume $\sigma \cap g.\sigma \neq 0$, being all of the cells disconnected, this means that $g.\sigma = \sigma$, but a map from a disk to a disk has to have one fixed point, and therefore $g0\mathrm{Id}$.
To prove that $\mathrm{cd}\Gamma < \infty$, we will construct from X' a finite dimensional contractible, free Γ-complex X (which gives us a finite free $\mathbb{Z}\Gamma$ resolution of $\mathbb{Z}$ which is exactly what we want). The construction goes as follows:
The underlying set of X is defined by

$$X = \hom_{\Gamma'}(\Gamma, X')$$

where Γ' acts on Γ by left translation and $\hom_{\Gamma'}(-,-)$ denotes the map in the category of left Γ'-sets. Since the right action of Γ on itself commutes with the left action of Γ' on Γ (it should remind of a bi-module condition), there is an induced (left) action of Γ on X, given by

$$(\gamma_0 \cdot f)(\gamma) := f(\gamma\gamma_0)$$

for $f \in X$, $\gamma, \gamma_0 \in \Gamma$. If we choose a set of coset representatives $\gamma_1, \gamma_2, \ldots, \gamma_n$ for Γ/Γ', then we obtain a bijection

$$\varphi \colon X \xrightarrow{\simeq} \prod_{i=1}^{n} X'$$

given by evaluation at $\gamma_1, \ldots, \gamma_n$ (The idea is we have to use $\hom_{\Gamma'}(\Gamma, X') = \hom_{\Gamma'}(\bigsqcup_{i=1}^{n} \Gamma', X') \simeq \prod_{i=1}^{n} \hom_{\Gamma'}(\Gamma', X') \simeq \prod_{i=1}^{n} X'$). Since the product on the right has a natural CW-structure, we can use φ to give X a topology and a CW-structure. This structure is independent of the choice of coset representatives; for if we replace $\gamma_1, \ldots, \gamma_n$ by $\gamma'_1\gamma_1, \ldots, \gamma'_n\gamma_n$ ($\gamma'_i \in \Gamma'$), then the new φ is obtained from the old one by composition with the CW-isomorphism:

$$\prod_{i=1}^{n} \gamma'_i \colon \prod_{i=1}^{n} X' \to prod_{i=1}^{n} X'$$

Notice that it is an isomorphism because Γ' act by isomorphisms. The structure is also independent of the ordering of the cosets. It follows that the Γ-action on X preserves the CW-structure. Indeed, for any $\gamma \in \Gamma$ we have the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "\gamma"] \arrow[dr, "\varphi'"'] & & X \arrow[dl, "\varphi"] \\
& \prod_{i=1}^{n} X' &
\end{tikzcd}$$

where φ is defined via the coset representatives $\{\gamma_i\}_i$ and φ' is defined via the coset representatives $\{\gamma_i\gamma\}_i$. Thus X is a well-defined Γ-complex, which is clearly contractible (product of contractible spaces) and finite dimensional.
To complete the proof that $\mathrm{cd}\Gamma < \infty$, we will show that Γ acts freely on X. There is a canonical map $X \to X'$, given by evaluation at $1 \in \Gamma$ (recall that points of X are functions $f \colon \Gamma \to X'$). This map is Γ'-equivariant and takes cells to cells. Since Γ' acts freely on X', it follows that Γ' acts freely on X. For any cell σ of X, then we have $\Gamma_\sigma \cap \Gamma' = \{1\}$, hence Γ_σ is finite. But Γ is torsion-free, so these finite isotropy groups Γ_σ are trivial (if a group is finite then it has to have torsion, so the only possibility for a finite torsion-free group is to be trivial). $\square$

5.2.3 Resolution of Finite Type

Our next goal is to study a different sort of finiteness condition, where we require that there be a projective resolution P with each P_i finitely generated. In this section we collect some general facts about such resolutions over an arbitrary ring R. Then in the next two subsections we specialize to resolutions of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.
We begin by reviewing the theory of finitely presented modules:

Proposition 5.2.5. *The following conditions on an R-module M are equivalent:*

- *There is an exact sequence $R^m \to R^n \to M \to 0$ for some integers m, n*
- *There is an exact sequence $P_1 \to P_0 \to M \to 0$ for some finitely generated projectives P_0, P_1*

- *M is finitely generated, and for every surjection $\epsilon\colon P \to M$, with P finitely generated and projective,* $\ker\epsilon$ *is finitely generated.*

The proof is based on *Schanuel's Lemma*

Lemma 5.2.4. *Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ be exact sequences with P, P' projective. Then $P \oplus K' \simeq P' \oplus K$*

Proof. Let Q be the pullback if the given maps

$$\begin{array}{ccc} & & P' \\ & & \big\downarrow{\scriptstyle \pi'} \\ P & \xrightarrow{\pi} & M \end{array}$$

i.e. Q is the submodule of $P \times P'$ consisting of those pairs (x, x') such that $\pi(x) = \pi'(x')$. One the verify easily that there is a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K' \arrow[r, "\mathrm{Id}"] \arrow[d] & K' \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "\mathrm{Id}"] & Q \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Since P and P' are projective, the two exact sequences involving Q must split, yielding $P \oplus K' \simeq Q \simeq P' \oplus K$. $\square$

Proof. of 5.2.5. Clearly (3) $\Rightarrow$ (1) In fact we know that M is finitely generated, hence there is a surjection $R^n \to M \to 0$ for some n, call the map π. Now we know that $\ker(\pi)$ is finitely generated, so we have a sequence $R^m \to \ker(\pi) \to R^n$. Just attach these two sequence to obtain (1). (1) $\Rightarrow$ (2) is trivial. To prove (2) $\Rightarrow$ (3), note first that M is certainly finitely generated if (2) holds, since P_0 is finitely generated. Now apply Schaunel Lemma to get $P \oplus \ker\{P_0 \to M\} \simeq P_0 \oplus \ker\epsilon$. The lefthand side being finitely generated by hypothesis, it follows that $P_0 \oplus \ker\epsilon$ is finitely generated, hence so is $\ker\epsilon$. $\square$

M is said to be finitely presented if the conditions of Proposition 5.2.5 hold. An exact sequence as in (1) is said to give a finite presentation of M with n generators and m relations. It is natural to generalize finite presentation as follows: A resolution or partial resolution (P_i) is said to be of *finite type* if each P_i is finitely generated. A module M is said to be of type FP_n ($n \geq 0$) if there is a partial projective resolution $P_n \to \cdots \to P_0 \to M \to 0$ of finite type. Thus the FP_0 condition is simply finite generation, FP_1 is finite presentation, and the conditions $FP_2, FP_3, \ldots$ are successive strengthenings of finite presentation.
We can generalize 5.2.5 in the following way

Proposition 5.2.6. *For any module M and integer $n \geq 0$ the following conditions are equivalent:*

1. *There is a partial resolution $F_n \to \cdots \to F_0 \to M \to 0$ with each F_i free of finite rank*
2. *M is of type FP_n*

3. *M is finitely generated, and for every partial projective resolution $P_k \to \cdots \to P_0 \to M \to 0$ of finite type with $k < n$, $\ker\{P_k \to P_{k-1}\}$ is finitely generated*

Proof. See Brown's book at page 193 prop. 4.3 □

We will primarily interested in the case where the conditions of 5.2.6 hold for all integers $n \geq 0$. This situation is characterized as follows:

Proposition 5.2.7. *The following conditions on a module M are equivalent:*

1. *M admits a free resolution of finite type*
2. *M admits a projective resolution of finite type*
3. *M is of type FP_n for all integers $n \geq 0$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) trivially. If (3) holds then we can use 5.2.6.3 to construct a free resolution of finite type step by step, so (3) $\Rightarrow$ (1). □

We say that M is of type FP_∞ if these conditions hold.
We close this subsection by mentioning some useful formal properties which the homology and cohomology functors $\mathrm{Tor}_*^R(M,-)$ and $\mathrm{Ext}_R^*(M,-)$ satisfy when M is of type FP_n. Note first that the homology functors $\mathrm{Tor}_*^R(M,-)$ always commute with direct limits, in the following sense: Let $\{N_i\}_{i\in I}$ be a direct system of R-modules, where I is a directed set , and let $N = \varinjlim_{i\in I} N_i$. Thus N is the universal target of compatible family of maps $N_i \to N$. These maps induce a compatible family of abelian group homomorphisms $\mathrm{Tor}_*^R(M,N_i) \to \mathrm{Tor}_*^R(M,N)$, from which we obtain a map

$$\varphi\colon \varinjlim_{i\in I} \mathrm{Tor}_*(M,N_i) \to \mathrm{Tor}_*(M,N)$$

the assertion, then, is that φ is an isomorphism. This follows directly from the the definition of $\mathrm{Tor}_*^R(M,-)$ as $H_*(P\otimes_R -)$, where P is a projective resolution of M, together with the following two facts:

(a) $P \otimes_R -$ commutes with direct limits

(b) $H_*(-)$ commutes with direct limits

Functors of the form $\mathscr{H}_R(P,-)$ however, do not in general commute with direct limit, so we cannot expect $\mathrm{Ext}_R^*(M,-)$ to commute with direct limits. But we can prove that this does hold under suitable FP_n hypothesis. For simplicity, we will confine ourselves to the case $n = \infty$.

Proposition 5.2.8. *If M is of type FP_∞ then $Ext_R^*(M,-)$ commutes with direct limits.*

Surprisingly, these formal properties of $\mathrm{Ext}_R^*(M,-)$ and $\mathrm{Tor}_*^R(M,-)$ characterize the FP_∞ property. In fact, one can prove:

Theorem 5.2.2. *The following conditions are equivalent:*

1. *M is of type FP_∞*
2. *$Ext_R^*(M,-)$ commutes with direct limits*
3. *$Tor_*^R(M,-)$ commutes with direct products*

5.3 Groups of Type FP_n

We now specialize the theory of the precedent subsection ot the case $R = \mathbb{Z}\Gamma$, $M = \Gamma$. We will say that Γ is of type FP_n if $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}\Gamma$-module. Thus every group is of type FP_0, and it is easy to see that Γ is of type FP_1 if and only if it is finitely generated.

Proposition 5.3.1. *Let $\Gamma' \subseteq \Gamma'$ be a subgroup of finite index. Then Γ is of type FP_n $(0 \leq n \leq \infty)$ if and only if Γ' is of type FP_n*

Proof. Any (partial) projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of finite type can also be regarded as a (partial) projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$, and as such it is still of finite type since $[\Gamma : \Gamma'] < \infty$. This proves the only if part of the statement.
Conversely, suppose Γ' is of type FP_n. We will show that the $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ satisfies condition 5.2.6.3. Let P be a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ of finite type and length $k < n$. Regarding P as a partial resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$, we can apply 5.2.6.3 to conclude that $\ker\{P_k \to P_{k-1}\}$ is finitely generated over $\mathbb{Z}\Gamma'$. But then $\ker\{P_k \to P_{k-1}\}$ is certainly finitely generated over $\mathbb{Z}\Gamma$ so 5.2.6.3 holds for $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. $\square$

Finally we record for future references an important consequence of 5.2.8

Proposition 5.3.2. *Let Γ be a groups of type FP_∞ and let n be an integer such that $H^n(\Gamma, \mathbb{Z}\Gamma) = 0$. Then $H^n(\Gamma, F) = 0$ for all free $\mathbb{Z}\Gamma$-modules.*

Proof. If F is of finite rank then the result follows from the additivity of $H^n(\Gamma, -)$. In the general case, choose a basis $(e_i)_{i\in I}$ for F and note that $F = \varinjlim F_J$, where J ranges over the finite subsets of I and F_J is the submodule of F generated by the e_i for $i \in J$. Then F_J is free of finite rank; using 5.2.8 we conclude that $H^n(\Gamma, F) = \varinjlim H^n(\Gamma, F_J) = 0$ $\square$

5.4 Groups of Type FP and FL

We now combine the two types of finiteness conditions which we have considered in this chapter. A resolution is said to be *finite* if it is both of finite type and of finite length. A group Γ is said to be of type FP if $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}\Gamma$.

Proposition 5.4.1. *Γ is of type FP if and only if $cd\Gamma < \infty$ and Γ is of type FP_∞.*

Proof. We prove the easy part first. Assume Γ of type FP. Clearly, being the resolution of finite length, $\mathrm{cd}\Gamma < \infty$. Moreover by hypothesis we have a finite type projective resolution, therefore by Prop 5.2.7 Γ is of type FP_∞.
Conversely, if $\mathrm{cd}\Gamma$ is finite, and Γ is of type FP_∞ then we can construct a finite resolution as follows: Take a partial resolution $P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ of finite type, where $n = \mathrm{cd}\Gamma$, and let $P_n = \ker\{P_{n-1} \to P_{n-2}\}$. Then P_n is projective (Lemma 5.2.2) and finitely generated (Prop 5.2.6.3) so we have a finite projective resolution as claimed. $\square$

Note that it would have been enough in the proof above to assume that Γ was og type FP_n instead of FP_∞, where $n = \mathrm{cd}\Gamma$. It follows, for instance, that a finitely presented group Γ with $\mathrm{cd}\Gamma = 2$ is of type FP. Note also that the partial resolution $\{P_i\}_{i \leq n-1}$ above could have been taken free. Hence if Γ is of type FP then there is a finite projective resolution

$$0 \to P \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0 \tag{3}$$

which each F_i free. But there is no reason to expect to be able to take P free. Thus, for the first time since the beginning of these notes, there really seems to be a difference between what can be done with projective resolutions and what can be done using only free resolutions.
We are therefore led to introduce a still stronger finiteness condition

Definition 5.4.1. Γ is of type FL if $\mathbb{Z}$ admits a finite free resolution over $\mathbb{Z}\Gamma$.

It is obvious how to use topology to obtain examples of groups of type FL.

Proposition 5.4.2. *If there exists a $K(\Gamma, 1)$ which is a finite complex, then Γ is of type FL.*

The FP property also admits a topological interpretation, for which we need the following notion: A space Y is finitely dominated if there is a finite complex K such that Y is a retract of K in the homotopy category (i.e. we require maps $i\colon Y \to K$ and $r\colon K \to Y$ with $r \circ i \approx \mathrm{Id}_Y$).

Proposition 5.4.3. *If there exists a finitely dominated $K(\Gamma, 1)$, then Γ is of type FP.*

Proof. We only give a sketchy proof. Let Y be a $K(\Gamma, 1)$-complex dominated by a finite complex K. One can choose K so that the maps $Y \leftrightarrows K$ induce π_1-isomorphisms. Letting $\tilde{Y}$ and $\tilde{K}$ be the respective universal covers, one deduces that the cellular chain complex $C(\tilde{Y})$ is a retract of $C(\tilde{K})$ in the homotopy category of chain complexes over $\mathbb{Z}\Gamma$. Since $C(\tilde{Y})$ is a free resolution of $\mathbb{Z}$ and $C(\tilde{K})$ is finite free complex, it follows that $H^*(\Gamma, -)$ commutes with direct limits and that $H^i(\Gamma, -) = 0$ for $i > \dim K$. In view of 5.2.2, this implies that Γ is of type FP. □

Having carefully explained the difference between the FP condition and the FL condition, we are now forced to admit that there are no known examples of groups of type FP which are not FL.
In order to better appreciate the problem, let's see what the obstruction is to proving that a group of type FP is of type FL. Suppose Γ is of type FP, and choose a finite projective resolution as in 3 which is free except in the top dimension. The projective P which occurs in the top dimension might have the property that $P \oplus F$ is free for some free module F of finite rank. In this case P is said to be stably free, and we can modify 3 as in the proof of 5.2.4 to obtain a finite a finite free resolution. Conversely, if there exists a finite free resolution, then we can compare it to 3 via the generalization of Schaunel Lemma (See Brown, page 193 Lemma 4.4) to deduce that P is stably free. Thus we have

Proposition 5.4.4. *Let Γ be a group of type FP and let $0 \to P \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ be a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ with each F_i free. Then Γ is of type FL if and only if P is stably free*

Thus the question as t whether there exist groups of type FP which are not of type FL has led to a more fundamental question: Do there exist finitely generated projective which are not stably free? The surprisingly fact, however is that there are no known example with Γ torsion-free, and a group of type FP is necessarily torsion-free by 5.2.1.

Proposition 5.4.5. *Let Γ be a torsion-free group and Γ' a subgroup of finite index. Then Γ is of type FP if and only if Γ' is of type FP.*

Proof. This follows from 5.2.1 and 5.3.1 □

In particular, if Γ is torsion-free group which contains a subgroup of finite index which is of type FL, then we know from 5.4.5 that Γ is of type FP, even though we don't know that Γ is of type FL.
We close this subsection by discussing some special features of the top-dimensional cohomology of a group of type FP. First, we note the following improvement of 5.2.2

Proposition 5.4.6. *If Γ is of type FP then $cd\Gamma = \max\{n \mid H^n(\Gamma, \mathbb{Z}\Gamma) \neq 0\}$*

Proof. This follows from 5.2.2 and 5.3.2 □

Recall that the cohomology groups $H^i(\Gamma, \mathbb{Z}\Gamma)$ admit a canonical right Γ-module structure. For any left Γ-module M, we can therefore form the tensor product $H^i(\Gamma, \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} M$, and there is a canonical map

$$\varphi\colon H^*(\Gamma, \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} M \to H^*(\Gamma, M)$$

defined as follows on the cochain level: let P be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$; given a cochain $u \in \mathscr{H}_\Gamma(P, \mathbb{Z}\Gamma)$ and an element $m \in M$, we send $u \otimes m$ to the cochain $x \mapsto u(x)m$, where $x \in P$, in $\mathscr{H}_\Gamma(P, M)$.
We can now state the following *universal coefficient theorem* for the top-dimensional cohomology of a group of type FP:

Proposition 5.4.7. *If Γ is of type FP and $n = cd\Gamma$, then*

$$\varphi\colon H^*(\Gamma, \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma} M \to H^*(\Gamma, M)$$

is an isomorphism for all Γ-modules M.

5.5 Topological Interpretation

We have seen that the existence of a finite dimensional $K(\Gamma, 1)$ complex implies that Γ has finite cohomological dimension. Similarly 5.4.2 and 5.4.3 show the existence of a finite (resp. finitely dominated) $K(\Gamma, 1)$ implies that Γ is of type FL (resp. FP). The purpose of this subsection is to consider the converse implications. We also want to give a topological interpretation of the Γ-modules $H^*(\Gamma, \mathbb{Z}\Gamma)$. We will require a tiny bit of homotopy theory, namely, the Hurewicz theorem.
The following theorem is due to Eilenberg-Ganea and Wall.

Theorem 5.5.1. *Let Γ be an arbitrary group and let $n = \max\{cd\Gamma, 3\}$. Then there exists an n-dimensional $K(\Gamma, 1)$-complex Y. If Γ is finitely presented and of type FL (resp. FP) then Y can be taken to be finite (resp. finitely dominated)*

We allow here the possibility of $\text{cd}\Gamma = \infty$, in which case the theorem simply asserts the existence of a $K(\Gamma, 1)$-complex. As an immediate consequence of 5.5.1 we have:

Corollary 5.5.1. *If $cd\Gamma \geq 3$ then $cd\Gamma = geom\ dim\Gamma$.*

Of course we also have $\text{cd}\Gamma = \text{geom dim}\Gamma$ if $\text{cd}\Gamma = 0$ (since Γ is trivial) or if $\text{cd}\Gamma = 1$ (by a theorem of Stallings-Swan). So in view of Theorem 5.5.1 the only exception would be $\text{cd}\Gamma = 2$ and $\text{geom dim}\Gamma = 3$ (the theorem builds a 3-dim $K(\Gamma, 1)$). It is not known whether this possibility can actually occur.

Proof of 5.5.1. We will construct the skeleta Y^k of the desired Y inductively. To start the induction, let Y^2 be the 2-complex associated to some presentation of Γ. Thus $\pi_1 Y^2 \simeq \Gamma$. If Γ is finitely presented, Y^2 can be taken to be finite. Note that its universal cover X^2 has $H_i = 0$ for $0 < i < 2$.
Now assume inductively that Y^{k-1} has been constructed, and that its universal cover X^{k-1} has $H_i = 0$ for $0 < i < k-1$. If Γ is finitely presented and of type FP, assume further that Y^{k-1} is finite. Choose a set of generators (z_α) for the Γ-module $H_{k-1}X^{k-1}$ (It's generated by the cells as a $\mathbb{Z}$-module). By the Hurewicz theorem we can find for each α a map $f_\alpha\colon S^{k-1} \to X^{k-1}$ which represents z_α, in the sense that $H_{k-1}(f_\alpha)\colon H_{k-1}S^{k-1} \to H_{k-1}X^{k-1}$ sends a generator of $H_{k-1}S^{k-1}$ to z_α.
We now set $Y^k = Y^{k-1} \cup \bigcup_\alpha e^k_\alpha$ where the k-cell e^k_α is attached to Y^{k-1} via the composite

$$S^{k-1} \xrightarrow{f_\alpha} X^{k-1} \xrightarrow{p} Y^{k-1}$$

Letting X^k be the universal cover of Y^k, we must verify that $H_i X^k = 0$ for $0 < i < k$. Note first that we can view X^{k-1} as the $(k-1)$-skeleton of X^k; indeed, X^k is obtained from X^{k-1} by attaching k-cells via the maps f_α[2] and their transforms under the action of Γ on X^{k-1}. It

[2] we are lifting $p \circ f_\alpha$!

is clear, then, that $H_iX^k = H_iX^{k-1}$ for $0 < i < k-1$, because using cellular homology it's immediate noticing that attaching high dimensional cells doesn't change the lower dimensional homology. In order to prove that $H_{k-1}X^k = 0$, consider the following piece of the l.e.s of the pair (X^k, X^{k-1})

$$H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}X^{k-1} \to H_{k-1}X^k \to H_{k-1}(X^k, X^{k-1})$$

by definition of cellular homology we have immediately that $H_{k-1}(X^k, X^{k-1}) = 0$. So we have the exact sequence

$$H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}X^{k-1} \to H_{k-1}X^k \to 0$$

It will therefore suffice to show that ∂ is surjective.
Recall that (by definition of cellular homology) $H_k(X^k, X^{k-1})$ (which is simply $C_k^{\text{cell}}(X^k)$, the k-th cellular chain group) is a free $\mathbb{Z}\Gamma$-module with one basis element for each k-cell of Y^k, i.e., for each index α. Explicitly, there is a basis (v_α) obtained as follows: if $\chi_\alpha\colon (D^k, S^{k-1}) \to (X^k, X^{k-1})$ is a characteristic map for the cell attached via f_α, then $v_\alpha \in H_k(X^k, X^{k-1})$ is defined to be the image under

$$H_k(\chi_\alpha)\colon H_k(D^k, S^{k-1}) \to H_k(X^k, X^{k-1})$$

of a generator of $H_k(D^k, S^{k-1}) \simeq \mathbb{Z}$. In view of the diagram

$$\begin{array}{ccc} H_k(D^k, S^{k-1}) & \xrightarrow[\simeq]{\partial} & H_{k-1}(S^{k-1}) \\ {\scriptstyle H_k(\chi_\alpha)}\downarrow & & \downarrow{\scriptstyle H_{k-1}(f_\alpha)} \\ H_k(X^k, X^{k-1}) & \xrightarrow{\partial} & H_{k-1}X^{k-1} \end{array}$$

it follows that $\partial v_\alpha = z_\alpha$ (assuming that the generators of $H_k(D^k, S^{k-1})$ and $H_{k-1}(S^{k-1})$ have been chosen compatibly), so that ∂ is indeed surjective.
Note for future reference that if $H_{k-1}X^{k-1}$ happens to be a *free* $\mathbb{Z}\Gamma$-module with basis (z_α), then $\partial\colon H_k(X^k, X^{k-1}) \to H_{k-1}X^{k-1}$ is an isomorphism. It then follows form the long exact sequence in homology of the pair (X^k, X^{k-1}) that $H_i(X^k) = 0$ for all $i > 0$, so that Y^k is a $K(\Gamma, 1)$.
To complete the inductive step, we must show that Y^k can be taken to be finite if Γ is finitely presented and of type FP, i.e., we must show in this case that $H_{k-1}(X^{k-1})$ is a finitely generated Γ module, in order to have to add only finitely many cells to kill the homology group.
To see this, we need only note that the cellular chain complex

$$C_{k-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

of X^{k-1} is a partial free resolution of *finite type*, since Y^{k-1} was assumed to be finite. Therefore

$$H_{k-1} = \ker\{C_{k-1} \to C_{k-2}\}$$

is finitely generated by Prop. 5.2.6 since Γ is of type FP.
Now we need to discuss how we can *stop* this inductive process at the right dimension, in order to show the finiteness conditions of the statement.
If $n = \infty$, we now continue this inductive process indefinitely, and

$$Y = \bigcup_k Y^k$$

is the desired $K(\Gamma, 1)$. If $n < \infty$, consider X^{n-1} (this makes sense because $n - 1 \geq 2$). Its cellular chain complex

$$C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

is a partial free resolution of length $n-1$, hence Lemma 5.2.2 implies that $H_{n-1}X^{n-1}$ is a projective $\mathbb{Z}\Gamma$-module. By the Eilenberg Trick (Lemma 5.2.3) there is a free module F such that $H_{n-1}X^{n-1} \oplus F$ is free. The goal now is to build a n-dimensional CW complex $\bar{X}$ such that $H_{n-1}(\bar{X}^{n-1})$ is free in order to apply the addendum we show before in the proof.
To this end, we replace Y^{n-1} by $\bar{Y}^{n-1} := Y^{n-1} \vee S^{n-1} \vee S^{n-1} \vee \cdots$, where there is one copy of S^{n-1} for each basis element of F. The effect of this on $C(Y^{n-1})$ is simply to add F to C_{n-1}, with $\partial_{|F} = 0$. We claim now that the universal cover $\bar{X}^{n-1}$ of $\bar{Y}^{n-1}$ has $H_{n-1}(\bar{X}^{n-1})$ free $\mathbb{Z}\Gamma$-module. In fact, $\bar{X}^{n-1} \cong X^{n-1} \bigvee_J S^{n-1}$ where J runs over the elements of the chosen basis of F and the action of Γ (for any such basis element there is a lift for each element in Γ). Now it's easy to notice that as a $\mathbb{Z}\Gamma$ module, $H_{n-1}(\bar{X}^{n-1}) \cong H_{n-1}(X^{n-1}) \oplus F$ and hence is free as claimed. We may therefore attach n-cells e^n_α to $\bar{Y}^{n-1}$ corresponding to basis elements (z_α) of $H_{n-1}\bar{X}^{n-1}$ and consider $\bar{X}^n$. We claim that As remarked above, the resulting $\bar{Y}^n = \bar{Y}^{n-1} \cup \bigcup e^n_\alpha$ will then be an n-dimensional $K(\Gamma, 1)$.
Suppose now that Γ is finitely presented and of type FL. Then Y^{n-1} is finite and the projective $H_{n-1}X^{n-1}$ is finitely generated. We know from 5.4.4 that $H_{n-1}X^{n-1}$ is stably free, so that there is a free module F of *finite rank* such that $H_{n-1}X^{n-1} \oplus F$ is free of finite rank. We now proceed as in the previous paragraph, and the resulting $\bar{Y}^n$ will be a finite $K(\Gamma, 1)$.
Finally suppose that Γ is finitely presented but only of type FP instead of FL. On the one hand, the general inductive step above gives us a finite complex

$$Y^n = Y^{n-1} \cup e^n \cup \cdots \cup e^n$$

whose universal cover has $H_i = 0$ for $0 < i < n$. Hence $\pi_i Y^n \overset{p_*}{\simeq} \pi_i X^n = 0$ for $1 < i < n$. On the other hand, we know that there is a $K(\Gamma, 1)$ of the form

$$\bar{Y}^n = Y^{n-1} \vee S^{n-1} \vee \cdots \cup e^n \cup \cdots$$

so that $\pi_i \bar{Y}^n = 0$ for all $i > 1$. We claim that Y^n dominates $\bar{Y}^n$. Indeed, the required maps

$$\bar{Y}^n \xrightarrow{i} Y^n \xrightarrow{r} \bar{Y}^n$$

with $ri \simeq \mathrm{Id}$ are easily constructed as follows: i and r are both defined to be the identity on the common subcomplex Y^{n-1}, and they are extended arbitrarily to the cells that were attached. (These extensions exists trivially for each e^n that was wedged onto Y^{n-1} in forming $\bar{Y}^n$, and they exists for each S^{n-1} because $\pi_{n-1} Y^n = 0$ and $\pi_{n-1} \bar{Y}^n = 0$). Finally, the homotopy $ri \simeq \mathrm{Id}$ is defined to be the constant homotopy on Y^{n-1} and is extended to all of $\bar{Y}^n$ by means of the vanishing of $\pi_{n-1} \bar{Y}^n$ and $\pi_n \bar{Y}^n$. □

For future reference we mention the following refinement of Theorem 5.5.1.

Corollary 5.5.2. *The complex Y in Theorem 5.5.1 can be taken to be a simplicial complex.*

Proof. Assume inductively that Y^{k-1} is simplicial, where the notation is that of the proof above. By simplicial approximation theorem, we may then take each attaching map $f_\alpha \colon S^{k-1} \to Y^{k-1}$ to be simplicial relative to some triangulation of S^{k-1}. The resulting space Y^k is then triangulable by a Theorem of Whitehead. □

Finally we give a topological interpretation of the right Γ-modules $H^*(\Gamma, \mathbb{Z}\Gamma)$, assuming that Γ is finitely presented and of type FL. We will need the following observation

Lemma 5.5.1. *Let Γ be a group and M a left Γ-module. Let $\hom_c(M, \mathbb{Z}) \subseteq \hom(M, \mathbb{Z})$ consists of all abelian group homomorphisms $f \colon M \to \mathbb{Z}$ such that, for every $m \in M$, $f(\gamma.m) = 0$ for all but finitely many $\gamma \in \Gamma$. Then there is a natural isomorphism*

$$\hom_\Gamma(M, \mathbb{Z}\Gamma) \simeq \hom_c(M, \mathbb{Z})$$

Moreover, this is an isomorphism of right Γ-modules, where Γ acts on $\hom_\Gamma(M, \mathbb{Z}\Gamma)$ via its right action on $\mathbb{Z}\Gamma$ and Γ acts on $\hom_c(M, \mathbb{Z})$ via its left action on M (i.e. $f \bullet \gamma(m) := f(\gamma.m)$).

Proof. A $\mathbb{Z}$-module map $F\colon M \to \mathbb{Z}\Gamma$ has the form

$$F(m) = \sum_{\gamma \in \Gamma} f_\gamma(m)\gamma$$

where $f_\gamma\colon M \to \mathbb{Z}$ and, for each $m \in M$, $f_\gamma(m) = 0$ for almost all $\gamma \in \Gamma$ (elements in $\mathbb{Z}\Gamma$ are finite formal sums!). One checks that such an F is a Γ-module homomorphism if and only if $f_\gamma(m) = f_1(\gamma^{-1}.m)$ for all $\gamma \in \Gamma$. We therefore have a map $\hom_\Gamma(M, \mathbb{Z}\Gamma) \to \hom_c(M, \mathbb{Z})$ given by $F \mapsto f_1$, and this map is an isomorphism with inverse

$$f \mapsto \{m \mapsto \sum_{\gamma \in \Gamma} f(\gamma^{-1}m)\gamma\}$$

The reader can easily verify that this isomorphism is natural and compatible with the right Γ-actions. □

Proposition 5.5.1. *If X is a contractible, free Γ-complex with compact quotient X/Γ, then there is an isomorphism*

$$H^*(\Gamma, \mathbb{Z}\Gamma) \simeq H^*_c(X; \mathbb{Z})$$

of right γ-modules, where the right action of Γ on $H^*_c(X, \mathbb{Z})$ is induced by the left action of Γ on X.

Proof. Suppose now that Γ is finitely presented and of type FL. By 5.5.1 there is a contractible, free Γ-complex X with X/Γ finite. Then $C(X)$ is a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, and $H^*(\Gamma, \mathbb{Z}\Gamma)$ is the cohomology of

$$\mathscr{H}_\Gamma(C(X), \mathbb{Z}\Gamma)$$

In view of the lemma, we have

$$\mathscr{H}_\Gamma(C(X), \mathbb{Z}\Gamma) \simeq \mathscr{H}_c(C(X), \mathbb{Z}) \subseteq \mathscr{H}(C(X), \mathbb{Z})$$

this isomorphism being compatible with the right Γ-action and the coboundary operators (the latter because of the naturality assertion in Lemma 5.5.1 so it induces an isomorphism after applying the cohomology functor.

Recall that $C(X)$ has a $\mathbb{Z}$-basis with one element for each cell σ of X. These basis elements are freely permuted by Γ and fall into finitely many orbits (because X/Γ is finite). It follows easily that $\mathscr{H}_c(C(X), \mathbb{Z})$ consists of those cochains $f \in \mathscr{H}(C(X), \mathbb{Z})$ such that $f(\sigma) = 0$ for all but finitely many cells σ: in fact $f \in \mathscr{H}_c(C(X), \mathbb{Z})$ if and only if for every σ, $f(\gamma.\sigma) = 0$ for all but finitely many γ's, but working with a finite set of representatives of each orbits leads directly to the fact that $f(\sigma) \neq 0$ only for finitely many cocycles. The cohomology of this complex is called the cohomology of X with *compact supports* and is denoted $H^*_c(X, \mathbb{Z})$. □

In view of 5.4.6 this yields:

Corollary 5.5.3. *If X is as in 5.5.1, then*

$$cd\Gamma = \max\{n \mid H^n_c(X; \mathbb{Z}) \neq 0\}$$

6 Further Topological Results

The purpose of this subsection is to see what finiteness properties of Γ can be deduced if we are given a $K(\Gamma, 1)$ which is a *manifold*. We will see many examples of this situation in the next subsection.

We begin with the analogue of 5.2.1 and 5.4.2 for $K(\Gamma, 1)$-manifolds:

Proposition 6.0.1. *Suppose Y is a d-dimensional $K(\Gamma,1)$-manifold (possibly with boundary).*

1. *$cd\Gamma \leq d$ with equality if and only if Y is closed.*
2. *If Y is compact, then Γ is of type FL.*

Proof. We present a sketchy proof only for the case Y is smooth.

1. If Y is smooth, using for example Whitehead's triangulation Theorem, it can be shown that Y has the homotopy type of a CW complex Y' of dimension $\leq d$, so $\mathrm{cd}\Gamma \leq d$ by 5.2.1. If Y is closed, then we have $H^d(\Gamma,\mathbb{Z}_2) \cong H^d(Y;\mathbb{Z}_2) = \mathbb{Z}_2 \neq 0$ and so $\mathrm{cd}\Gamma = d$. If Y is not closed, then one can deduce from Poincaré duality with local coefficients that $H^d(\Gamma,M) = H^d(Y;M) = 0$ for all Γ-modules M, so that $\mathrm{cd}\Gamma < d$.
2. If Y is compact, then we can take Y' to be finite, so Γ is of type FL by 5.4.2.

□

Next we wish to reinterpret $H^*(\Gamma,\mathbb{Z}\Gamma)$ (cf Prop. 5.5.1) in case there is a compact $K(\Gamma,1)$-manifold Y. Let X be the universal cover of Y. Since X is simply-connected, it is certainly orientable, and we denote by Ω its *orientation module*. Thus Ω is an infinite cyclic group whose two generators correspond to the two orientations of X. The action of Γ on X induces an action of Γ on Ω, with an element $\gamma \in \Gamma$ acting as a ± 1 according as the action of γ on X is orientation preserving or orientation reversing. Note that Γ acts trivially on Ω if and only if Y is orientable. Finally, we make the convention that the reduced homology of a space Z, denoted $\tilde{H}_*(Z)$, is the homology of the augmented chain complex of Z. In particular $\tilde{H}_{-1}(Z) = \mathbb{Z}$. We can now state:

Proposition 6.0.2. *Let Y be a compact d-dimensional $K(\Gamma,1)$-manifold (possibly with boundary). Let X be its universal cover and let Ω be the corresponding orientation module. Then there are Γ-module isomorphisms*

$$H^i(\Gamma,\mathbb{Z}\Gamma) \simeq \tilde{H}_{d-i-1}(\partial X) \otimes \Omega$$

For all i. In particular, if Y is a closed manifold, then

$$H^i(\Gamma,\mathbb{Z}\Gamma) \simeq \begin{cases} 0 & i \neq d \\ \Omega & i = d \end{cases}$$

Proof. Since Y is compact and has the homotopy type of of a finite complex, we have

$$H^i(\Gamma,\mathbb{Z}\Gamma) \simeq H^i_c(X)$$

by Prop. 5.5.1. On the other hand, there is a Poincaré-Lefschetz duality isomorphism

$$H^i_c(X) \simeq H_{d-i}(X,\partial X)$$

This however is not canonical; it depends on a choice of orientation of X. In particular, it commutes or anti-commutes with the action of an element $\gamma \in \Gamma$ according as γ preserves or reverses the orientation of X. Consequently, we have a Γ-module isomorphism

$$H^i_c(X) \simeq H_{d-i}(X,\partial X) \otimes \Omega$$

To see it use the naturality formula for the cap product and recall that we see cohomology as a right Γ-module). We tensor with the orientation module because depending on the orientation behaviour of $\gamma \in \Gamma$ we could have a minus sign popping out in the equality. Finally since $\tilde{H}_*(X) = 0$ the l.e.s of the pair gives

$$H_{d-i}(X,\partial X) \simeq \tilde{H}_{d-i-1}(\partial X)$$

The proposition follows at once. □

Corollary 6.0.1. *Under the hypothesis of Prop. 6.0.2, there exists at least one integer k such that $\tilde{H}_k(\partial X) \neq 0$. Moreover, letting*

$$l = 1 + \min\{k \mid \tilde{H}_k(\partial X) \neq 0\}$$

we have

$$cd\Gamma = d - l$$

Proof. Immediate from 6.0.2 and 5.4.6. Note that $l > 0$ if $\partial Y \neq \emptyset$. Thus this corollary makes more precise the inequality $\mathrm{cd}\Gamma < d$ of 6.0.1.1 in this case $\square$

6.1 Further Examples

The aim of this subsection is to study the cohomological dimension of $SL_n(\mathbb{Z})$ and more interestingly of certain subgroups of it. We will present explicit computation only for the case $n = 2$. Before starting we need some preliminary notions

6.1.1 $S_n(\mathbb{Z})$ is virtually torsion-free

Consider the group $SL_n(\mathbb{Z})$, for $n \geq 2$. We begin with the following observation

Proposition 6.1.1. *$SL_n(\mathbb{Z})$, for $n \geq 2$ has infinite cohomological dimension*

Proof. We will show that $SL_n(\mathbb{Z})$ has a torsion subgroup. To this end, consider the block matrix

$$J := \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

formed by the blocks $-\mathrm{Id}_2$ and Id_{n-2}. Clearly $J \in SL_n(\mathbb{Z})$ and $J^2 = \mathrm{Id}_n$. So by the fact that this torsion subgroup has infinite cohomological dimension we know that $\mathrm{cd}SL_n(\mathbb{Z}) = \infty$ as claimed $\square$

We know, however, that it has torsion-free subgroups Γ of finite index. Fix an integer $n \geq 1$. For any integer $N \geq 2$, let $\Gamma(N)$ be the kernel of the canonical map

$$SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/N\mathbb{Z})$$

i.e. $\Gamma(N) := \{g \in SL_n(\mathbb{Z}) \mid g \equiv 1 \pmod{N}\}$, where 1 denotes the identity matrix. The group $\Gamma(N)$ is called the *principal congruence subgroup of $SL_n(\mathbb{Z})$ of level N*. Note that $\Gamma(N)$ has finite index in $SL_n(\mathbb{Z})$, since $SL_n(\mathbb{Z}/N\mathbb{Z})$ is finite. We claim that $\Gamma(N)$ is torsion-free for $N \geq 3$. To prove this fact, we need a technical lemma before:

Lemma 6.1.1. *Let p be a fixed prime and let A be an $n \times n$ matrix of integers such that $A \equiv 1 \pmod{p}$. If $A \neq 1$; then there is a unique positive integer $d = d_p(A)$ such that*

$$A \equiv 1 \pmod{p^d} \quad A \not\equiv 1 \pmod{p^{d+1}}$$

Moreover, $d_p(A^q) = d(A)$ for any prime $q \neq p$. If p is odd or $d_p(A) \geq 2$, then $d_p(A^p) = d(A) + 1$

Proof. We first prove uniqueness. Let

$$\begin{aligned} A = 1 \pmod{p^{d_1}} \quad A \neq 1 \pmod{p^{d_1+1}} \\ A = 1 \pmod{p^{d_2}} \quad A \neq 1 \pmod{p^{d_2+1}} \end{aligned}$$

and assume $d_1 < d_2$ i.e. $d_1 + 1 \leq d_2$. This implies that if $p^{d_2} \mid A - 1$ then $p^{d_1+1} \mid A - 1$, which is an absurd. Therefore there exist at most one of such exponents.
Existence is clear by hypothesis, in the worst case taking $d = 1$ will do the job. As a notation convention, set $d_p(A)$ as the unique exponent for the matrix A modulo p with the properties listed in the statement of the lemma
We show now the claim about $d_p(A^q)$ for $p \neq q$. Let $A = 1 + p^{d_p(A)}B$ and consider

$$A^q = \left(1 + p^{d_p(A)}B\right)^q = 1 + \binom{q}{1}p^{d_p(A)}B + \binom{q}{2}p^{2d_p(A)}B^2 + \cdots + p^{qd_p(A)}B^q$$

Notice that clearly $A^q = 1 \pmod{p^{d_p(A)}}$ and $A^q \neq 1 \pmod{p^{d_p(A)+1}}$ because $p \not| qB$ (otherwise we apply the same reasoning made in the uniqueness statement). Now consider A^p. As before we have

$$A^p = \left(1 + p^{d_p(A)}B\right)^p = 1 + \binom{p}{1}p^{d_p(A)}B + \binom{p}{2}p^{2d_p(A)}B^2 + \cdots + p^{pd_p(A)}B^q$$

But this time $p \mid pB$ instead, so $A^p = 1 \pmod{p^{d_p(A)+1}}$ but $A^p \neq 1 \pmod{p^{d_p(A)+2}}$ because $p \not| B$ ($p^{d_p(A)+2}$ divides every addend but the first two). Therefore the claim. $\square$

Proposition 6.1.2. *$\Gamma(N)$ is torsion-free for $N \geq 3$*

Proof. First notice that, if $A \in \Gamma(N)$, then $A = 1 \pmod q$ for some prime $q \mid N$. By Lemma 6.1.1, there exists a unique $d_q(A)$ s.t. $A = 1 \pmod{q^{d_q(A)}}$ and $A \neq 1 \pmod{q^{d_q(A)+1}}$. Now assume that $A^p = 1$, for some prime p. The first case is $(p, N) = 1$, then by the lemma we proved before $1 = d_q(A^p) = d_q(A)$ for any prime q dividing N. This means $A = 1$. Now assume $p \mid N$. So for $(q = p)$ we have that $1 = d_p(A^p) = d_p(A) + 1$ and therefore $A = 1 \pmod{p^0}$ which is an equivalent way to say $A = 1$. $\square$

6.1.2 $SL_n(\mathbb{R})$ acts properly on a quotient of the space of positive definite quadratic forms

Definition 6.1.1 (Quadratic Form)**.** A Quadratic form on $\mathbb{R}^n$ is a function $Q\colon \mathbb{R}^n \to \mathbb{R}$ of the form

$$Q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$$

The matrix $A = (a_{ij})$ can be taken to be symmetric, and it is then uniquely determined by Q.

The form Q (and the matrix A) are called positive definite if $Q(x) > 0$ for all $x \neq 0 \in \mathbb{R}^n$. We now define X to be the space of positive definite quadratic forms on $\mathbb{R}^n$.

Proposition 6.1.3. *X is a contractible manifold of dimension $d = \frac{n(n+1)}{2}$*

Proof. Notice that for a $n \times n$ matrix A to be positive definite is equivalent to have all eigenvalues strictly greater than zero. Using the fact that we can simultaneously diagonalize two symmetric and positive definite matrices A and B, it's easy to show that the segment $tA + (1 - t)B$ is entirely contained in X. In fact diagonalize both of them via the same matrix Q, and connect the two diagonalized matrices via the segment and check that for every $t \in [0, 1]$, the segment lies in X. Now go back to the original case by the inverse of the diagonalization process.

Notice that this proves that X is contractible as well.
We are left to show that X is a manifold of dimension $d = \frac{n(n+1)}{2}$. We can prove both fact easily: first notice that X is an open subset of the space Y of symmetric matrices, and then recall that such space is a manifold of dimension $\frac{n(n+1)}{2}$. $\square$

There is a right action of $L = GL_n(\mathbb{R})$ on $\mathbb{R}^n$ by matrix multiplication, and this induces a left action of L on X, said to be given by change of variable

$$(g.Q)(x) = Q(x.g)$$

for $Q \in X$, $g \in L$, $x \in \mathbb{R}^n$. In terms of symmetric matrices, this action takes form

$$g.A = gAg^t$$

It is well know that any $Q \in X$ is equivalent under change of variable to the standard form $Q_0 = \sum x_i^2$, so the action of L on X is transitive. Moreover the isotropy group L_{Q_0} is the orthogonal group $K = O_n(\mathbb{R})$ (because it has to hold the relation $gg^t = \mathrm{Id}$). We therefore have a bijection

$$X \cong L/K$$

of left L-spaces, which can be shown to be an homeomorphism.

Proposition 6.1.4. *The action of L on X is proper, i.e. that the following condition is satisfied: For every compact set $C \subseteq X$, $\{g \in L \mid gC \cap C \neq 0\}$ is a compact subset of L.*

Proof. First notice that due to the fact that X is Hausdorff, its' equivalent to show that the map

$$\begin{aligned} \varphi_G \colon G \times X &\to X \times X \\ (g, x) &\mapsto (g.x, x) \end{aligned}$$

is proper[3] in the usual sense of proper function (i.e. preimage of compact subsets are compact). Now we show (using a little bit of point-set topology) that it's sufficient to show that preimages of points are compact.
Let $f : X \to Y$ be a continuous closed map, such that $f^{-1}(y)$ is compact (in X) for all $y \in Y$. Let K be a compact subset of Y. We will show that $f^{-1}(K)$ is compact.
Let $\{U_\lambda | \lambda \in \Lambda\}$ be an open cover of $f^{-1}(K)$. Then for all $k \in K$ this is also an open cover of $f^{-1}(k)$. Since the latter is assumed to be compact, it has a finite sub-cover. In other words, for all $k \in K$ there is a finite set $\gamma_k \subset \Lambda$ such that $f^{-1}(k) \subset \cup_{\lambda \in \gamma_k} U_\lambda$. The set $X \setminus \cup_{\lambda \in \gamma_k} U_\lambda$ is closed. Its image is closed in Y, because f is a closed map. Hence the set $V_k = Y \setminus f(X \setminus \cup_{\lambda \in \gamma_k} U_\lambda)$ is open in Y. It is easy to check that V_k contains the point k. Now $K \subset \cup_{k \in K} V_k$ and because K is assumed to be compact, there are finitely many points $k_1, \dots, k_s$ such that $K \subset \cup_{i=1}^s V_{k_i}$. Furthermore the set $\Gamma = \cup_{i=1}^s \gamma_{k_i}$ is a finite union of finite sets, thus Γ is finite. Now it follows that $f^{-1}(K) \subset f^{-1}(\cup_{i=1}^s V_{k_i}) \subset \cup_{\lambda \in \Gamma} U_\lambda$ and we have found a finite subcover of $f^{-1}(K)$, which completes the proof of the claim.
In order to complete the proof of the proposition, notice that the preimage $\varphi_G^{-1}(x, x)$ is $K \times \{x\}$ by construction, because K is the isotropy group. It's clearly compact and therefore the action is proper. $\square$

Notice now that the converse is trivially seen to be true as well, i.e. if an action is proper, then the isotropy groups are compact (take preimage of (x, x) via the proper map φ_G which is compact). So suppose now that G is a discrete subgroup of L. Then for any compact $C \subseteq X$,

[3] See Lee's Introduction to Topological Manifold, page 319, Prop 12.23

$\{g \in G \mid gC \cap C \neq 0\}$ is finite (compact subgroup of a discrete space). One deduces easily that the isotropy group G_x of any $x \in X$ is finite and that x has a neighbourhood U such that $g.U \cap U = \emptyset$ for $g \in G \setminus G_x$.
Finally suppose further that G is torsion-free. Then the finite isotropy groups G_x must be trivial, and it follows at once that the projection $X \to X/G$ is a regular covering map with group G. Thus X/G is a $K(G,1)$ and $H_*G \cong H_*(X/G)$.

Consider now the special linear group $SL_n(\mathbb{R})$. We can replace the space X we build before by a contractible manifold X_0 on which $SL_n(\mathbb{R})$ acts properly, with $\dim X_0 = \dim X - 1$. Namely, we take X_0 to be the quotient space of X obtained by identifying two quadratic forms which are (positive) scalar multiples of one another. One can verify that

$$X_0 \cong \frac{SL_n(\mathbb{R})}{SO_n(\mathbb{R})}$$

so that the $SL_n(\mathbb{R})$-action on X_0 is proper, because $SO_n(\mathbb{R})$ is compact. Moreover, X_0 is a contractible manifold whose dimension is $\frac{n(n+1)}{2} - 1$.

6.1.3 $SL_2(\mathbb{R})$ acts properly on $\mathbb{H}$

We note that if $n = 2$ the space X_0 can be identified with the upper half plane $\mathbb{H} \subset \mathbb{C}$, with $SL_2(\mathbb{R})$ acting by linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z := \frac{az+b}{cz+d}$$

Indeed, one checks that this defines a transitive action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ and that the isotropy group at $z = i$ is $SO_2(\mathbb{R})$

Proposition 6.1.5. *The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ as defined above is transitive and the isotropy group at $z = i$ is $SO_2(\mathbb{R})$*

Proof. For transitivity just consider the following composition of two elements of $SL_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$$

$$i \mapsto \frac{ai}{1/a} = a^2 i \mapsto \frac{a^2 i + x}{1} = a^2 i + x$$

Now we compute the isotropy group at i

$$\frac{ai+b}{ci+d} = i \Leftrightarrow ai + b = -c + di$$

implies $a = d$ and $b = -c$ and the determinant condition implies $a^2 + c^2 = 1$. So the general matrix which fix i is

$$\begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

which is the general form of an element of $SO_2(\mathbb{R})$ as claimed ∎

Putting together everything, this leads to the desired homeomorphism

$$\mathbb{H} \cong \frac{SL_2(\mathbb{R})}{SO_2(\mathbb{R})} \cong X_0$$

6.1.4 Cohomological dimension of Γ

It's now time to put everything together.
We give as a fact that the cohomological dimension of the group U of the $n \times n$ strict upper triangular matrices is

$$\mathrm{cd}U = \frac{n(n-1)}{2}$$

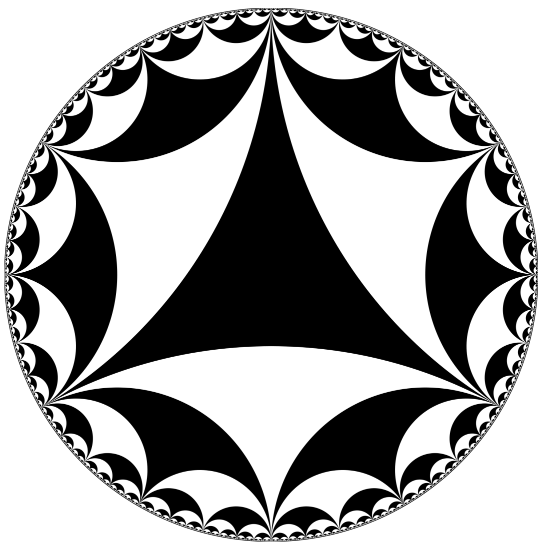
Note now that $\Gamma \cap U$ has finite index in U and hence has $\mathrm{cd} = n(n-1)/2$ by Prop. 5.2.3.1. In particular we just showed that

$$\mathrm{cd}\Gamma \geq \frac{n(n-1)}{2} \tag{4}$$

We are now ready to outline a proof, (explicit for $n = 2$) that Γ is of type FL and that equality holds in 4. At the end we will indicate a way on how to prove this result for any n using a generalization of what we are going to do now.
Let X be the space which we called X_0 in the subsection 6.1.2. Recall that X is the space of positive definite quadratic forms on $\mathbb{R}^n$, modulo multiplication by positive scalars. Recall that X is a contractible manifold of dimension without boundary $\frac{n(n+1)}{2} - 1$, that $SL_n(\mathbb{R})$ (hence also Γ) acts on X and that X/Γ is a $K(\gamma, 1)$. Recall also that if $n = 2$ then we can identify X with the upper half plane (or, equivalently the open unit disk) with $SL_2(\mathbb{R})$ acting by Möbius transformations. In view of 6.0.1.1 we have

$$\mathrm{cd}\Gamma \leq 2 = \frac{n(n+1)}{2} - 1$$

Moreover one can show that X/Γ is non compact, so that 6.0.1.1 implies that strict inequality holds above. But we can do substantially better than this by giving a direct geometric construction instead of relying on the generalities given in 6.0.1. Namely we will show that $XGamma$ admits a deformation retraction onto a subspace which is a CW-complex of dimension $1 = \frac{n(n-1)}{2}$. This will show that $\mathrm{cd}\Gamma \leq 1 = \frac{n(n-1)}{2}$ and hence will prove our claim that equality holds in 4.
We begin by describing how this is done for $n = 2$, using the disk model for $X \cong \mathbb{H}$.
There is a tessellation of X by *Ideal hyperbolic triangles* which is compatible with the action of $SL_2(\mathbb{Z})$. It is obtained by starting with a single ideal triangle and let (i.e. a hyperbolic triangle with vertices on the unit circles) and generating further triangles by successive reflections across the sides.



The vertices of the triangles of the tiling are called cusps, and we denote by X^* the space obtained from X by adjoining the cusps; it can be viewed as a simplicial complex with a simplicial $SL_2(\mathbb{Z})$-action. [Note that we give X^* the usual simplicial topology, rather than the topology it inherits as a subset of the plane. In particular, the set ∂X^* of cusps is a discrete set in the simplicial topology. It can be shown, however, that the simplicial topology agrees with the usual topology on the open subspace X of X^*].
Now let T be the simplicial complement of ∂X^* in the barycentric sub-division K of X^*, i.e. T is the largest subcomplex of K disjoint from ∂X^*. (Explicitly, T consists of all simplices of K none of whose vertices are in ∂X^*.)

I claim that this simplicial complement T is, in a canonical way, a deformation retract of the geometric complement $X = X^* \setminus \partial X^*$. Indeed, one deforms X to T by pushing away from ∂X^* along straight lines (in the hyperbolic sense). More precisely, any $x \in X$, any $x \in X$ lies in a closed 2-simplex σ of K with one vertex v in ∂X^* and the opposite face $\tau \in T$. Since $x \neq v$, there is a well-defined ray from v to x, and the deformation moves x along this ray away from v until it hits τ.
Note that T and the deformation are described purely in terms of the simplicial structure on X^*, so they are compatible with the action of $SL_2(\mathbb{Z})$ and its subgroup Γ. It follows that T/Γ is a deformation retract of X/Γ, and it has dimension $1 = \frac{2(2-1)}{2}$ as required.

The generalization to $SL_2(\mathbb{Z})$ for arbitrary n is based on a theory due to Voronoj. Voronoj constructs an enlargement X^* of X, obtained by adjoining certain positive *semi*-definite quadratic forms. He gives X^* an explicit decomposition into convex cells which are permuted by the action of $SL_n(\mathbb{Z})$. The subspace $\partial X^* = X^* \setminus X$ is a subcomplex. The cells of X are not necessarily simplices, but X^* admits a barycentric subdivision K which is simplicial and which inherits a simplicial action of $SL_n(\mathbb{Z})$. It follows, as above, that $X = X^* \setminus \partial X^*$ admits a deformation retract (compatible with the $SL_n(\mathbb{Z})$ action onto the simplicial complement T of ∂X^* in K. On sees by looking at Voronoj's construction that ∂X^* contains the entire $(n-2)$-skeleton of X^*, and it follows easily that T has codimension $\geq n-1$. Thus

$$\dim T/\Gamma = \dim T \leq \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2}$$

as required. Finally, Voronoj in addition that X^* has only finitely many cells mod the $SL_n(\mathbb{Z})$-action, so T/Γ is in fact a finite complex and Γ is of type FL.