Stable classification of 4-manifolds and signature related questions

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The aim of this brief seminar will be to introduce the topic of my master thesis: "Stable Classification of certain families of four-manifolds" and how we can obtain useful insight about the signature of a manifold as a by-product of this classification.

1 Basic definitions and tools

Let us begin with the key definitions and results about stable classification:

Definition 1.0.1. Two manifolds are called *stably diffeomorphic* if, after choosing two natural numbers n, m, the new manifolds obtained by stabilization with copies of $S^2 \times S^2$ are diffeomorphic. In symbols:

$$M\#m(S^2 \times S^2) \cong N\#n(S^2 \times S^2)$$

We will make use of the notation $M \cong_s N$ to denote that the two manifolds are stably diffeomorphic.
At this point, it is natural for the reader to ask herself (or himself) why we should care about stable diffeomorphism.

- It's easy to notice that stable diffeomorphism is an equivalence relation and that if $M \cong N$, then $M \cong_s N$. If we relax this hypothesis to a mere homotopy equivalence the two notions are independent: in my thesis I mostly found examples of *(oriented) homotopy equiv* implies *stable diffeomorphism* but in Prof. Teichner's PhD Thesis [4], there is a description of a stable diffeomorphism invariant which is not an homotopy invariant. The reason of this independence lies in the independence between the notion of B-bordism and homotopy equivalence (think about the notion of oriented bordism).

- Stable diffeomorphism classification is, *in theory*, computable: there exists a procedure which let you reach a conclusion. The catch is that there are Spectral Sequences involved, so it's not always easy to conclude something, nevertheless something can always be said.
- there are results (in **Top**!) which let you pass from stable homeomorphism to homeomorphism under certain circumstances. We won't discuss them here because they are very technical.

We work in the smooth category because when dealing with stable classification, going from the smooth category to the topological category is just a matter of adding the Kirby-Siebenmann invariant $e \in H^4(M;\mathbb{Z}_2)$ to the result when needed (See [3] page 24 for a discussion of this issue).
Let us going on in the introduction of the necessary notions:

Definition 1.0.2. Let M be an oriented closed manifold of dimension n. Its *normal k-type* is a fibration over BSO, denoted with $\xi\colon B \to BSO$ through which the classifying map of the stable normal bundle $\nu_M\colon M \to BSO$ factors as follows

$$\begin{array}{ccc} & & B \\ & \overset{\tilde{\nu}_M}{\nearrow} & \downarrow \xi \\ M & \xrightarrow[\nu_M]{} & BSO \end{array}$$

such that

1. $(\tilde{\nu_M})_*\colon \pi_i(M) \to \pi_i(B)$ is an isomorphism for $i < k+1$ and an epimorphism for $i = k+1$.
2. $(\xi_k)_*\colon \pi_i(B) \to \pi_i(BSO)$ is an isomorphism for $i > k+1$ and a monomorphism for $i = k+1$.

Existence and uniqueness of the normal 1-type follows from the theory of Moore-Postnikov decompositions. Note that the normal 1-type is an invariant of stable diffeomorphism since $S^2 \times S^2$ has trivial stable normal bundle.

It turns out (See [4] or [3]) that the normal 1-type of a 4-manifold M only depends on the fundamental group $\pi_1(M)$ and the spin properties of M and $\tilde{M}$. We will call M a totally non-spin manifold if $\tilde{M}$ is not spin and almost spin if M is **not** spin but $\tilde{M}$ is spin.

Proposition 1.0.3. *Let π be a group, let M be an almost spin 4-manifold and let $c\colon M \to K(\pi,1)$ a map inducing an isomorphism on fundamental groups. Then there exists a unique element $w \in H^2(K(\pi,1);\mathbb{Z}_2)$ such that $c^*w = w_2(M)$.*

Proof. We have the following exact sequence:

$$0 \to H^2(K(\pi,1);\mathbb{Z}/2) \xrightarrow{c^*} H^2(M;\mathbb{Z}/2) \xrightarrow{p^*} H^2(\widetilde{M};\mathbb{Z}/2)$$

via a Serre SS argument or a Blakers Massey argument. The first map is injective and since by assumption $p^*w_2(M) = w_2(\widetilde{M}) = 0$ there exists a pre-image w of $w_2(M)$, unique due to injectiviness of c^*. □

The association of the class $w \in H^2(\pi;\mathbb{Z}_2)$ is sometimes called the w-type of the manifold M denoted by w_M. Notice that this association is defined up to an element of $\mathrm{Aut}(\pi)$, since different choice of c might yield different w. We will take care of this indeterminacy later.

Proposition 1.0.4. *The normal 1-type of a totally non-spin manifold with fundamental group π is given by*

$$\xi\colon K(\pi,1) \times BSO \xrightarrow{\pi_2} BSO$$

Proof. This is Lemma 3.1 in [3] page 9. For future references, we will recall the following commutative diagram, showing explicitly the lifting:

$$\begin{array}{ccc} & & K(\pi,1)\times BSO \\ & \overset{c\times\nu_M}{\nearrow} & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow{\nu_M} & BSO \end{array}$$

where $c\colon M \to K(\pi,1)$ is a map which classifies the universal cover of M or equivalently induces an isomorphism on fundamental groups. ν_M is clearly the map classifying the stable normal bundle of M. □

Proposition 1.0.5. *The normal* 1*-type of a spin manifold with fundamental group* π *is given by*

$$\xi\colon K(\pi,1)\times BSpin \xrightarrow{\gamma\circ\pi_2} BSO$$

where γ *is the map* $BSpin \to BSO$ *induced by the* 2*-fold covering* $Spin \to SO$*.*

Proof. This is lemma 3.5 in [3] page 11. Again, for the details we refer to the original paper. For the sake of completeness we recall the explicit lifting:

$$\begin{array}{ccc} & & K(\pi,1)\times BSpin \\ & \overset{c\times\tilde{\nu}_M}{\nearrow} & \downarrow{\scriptstyle \gamma\circ\pi_2} \\ M & \xrightarrow{\nu_M} & BSO \end{array}$$

where $c\colon M \to K(\pi,1)$ is the classifying map of the universal cover of M and $\tilde{nu}_M\colon M \to BSpin$ is the choice of a spin structure for the normal bundle of M. □

Proposition 1.0.6. *The normal* 1*-type of an almost-spin, orientable* 4*-manifold with fundamental group* π *and with* $w \in H^2(K(\pi,1);\mathbb{Z}_2)$ *s.t.* $c^*w = w_2(M)$ *is given by:*

$$Bp \oplus \eta_w\colon BSpin \times K(\pi,1) \to BSO$$

where $p\colon Spin \to SO$ *is the universal covering and* η_w *is a (stable) vector bundle over* $K(\pi,1)$ *s.t.* $w_1(\eta_w) = 0$ *and* $w_2(\eta_w) = w$*.*

Proof. This is Lemma 3.20, page 18 in [3]. Denote the bundle over M given by $\nu_M \oplus c^*(-\eta_w)$ by $\nu(\eta_w)$, where $-\eta_w$ is the stable inverse of η_w. By construction the bundle $\nu(\eta_w)$ has a spin structure, therefore we have a lift of its classifying map $\tilde{\nu}(\eta_w)\colon M \to BSpin$. The factorization is given as follows:

$$\begin{array}{ccc} & & K(\pi,1)\times BSpin \\ & \overset{c\times\tilde{\nu}(\eta_w)}{\nearrow} & \downarrow{\scriptstyle Bp\oplus\eta_w} \\ M & \xrightarrow{\nu_M} & BSO \end{array}$$

□

We are now ready to enunciate the main theorem of my work:

Theorem 1.0.7 (Kreck)**.** *The stable diffeomorphism classes of* 4*-manifolds with normal* 1*-type* ξ *are in one-to-one correspondence with* $\Omega_4(\xi) \big/ Aut(\xi)$

So we need a way to compute $\Omega_4(\xi)$. In most of the cases, an Atyiah-Hirzebruch Spectral Sequence is enough, for example in the totally non spin and spin cases. When dealing with the almost spin case, things get rather complicated. In certain cases (for example the one we are considering) we can still use the AHSS, but there is another spectral sequence which was invented to solve this problem:

Theorem 1.0.8. *Let h be a generalized homology theory which is connected, i.e. $\pi_i(h) = 0$ for $i < 0$. Furthermore, let $F \to B \xrightarrow{f} K$ be an h-orientable fibration and $\xi \colon B \to BSO$ a stable vector bundle.*

$$\begin{array}{ccccc} F & \longrightarrow & B & \overset{f}{\twoheadrightarrow} & K \\ & & \big\downarrow{\scriptstyle \xi} & & \\ & & BSO & & \end{array}$$

Then there exists a spectral sequence

$$E^2_{p,q} \cong H_p(K; h_q(M(\xi|F))) \Rightarrow h_{p+q}(M\xi)$$

Proof. We won't give details of its construction here, the interested reader might want to look at [2], page 748. □

It's really important to stress the existence of a so called, universal example for almost spin manifolds, which will turn out to be really important:

$$\begin{array}{ccccc} BSpin & \longrightarrow & BSO & \overset{w_2(\gamma)}{\twoheadrightarrow} & K(\mathbb{Z}_2, 2) \\ & & \Vert & & \\ & & BSO & & \end{array}$$

where γ is the canonical bundle over BSO. We called it universal because for any normal 1-type $Bp \oplus \eta_w \colon BSpin \times K(\pi, 1) \to BSO$, we can fit it in the following diagram:

$$\begin{array}{ccccc} BSpin & \xrightarrow{i} & BSpin \times K(\pi,1) & \overset{\pi_2}{\twoheadrightarrow} & K(\pi,1) \\ \Vert & & \big\downarrow{\scriptstyle Bp\oplus\eta_w} & & \big\downarrow{\scriptstyle w} \\ BSpin & \longrightarrow & BSO & \underset{w_2(\gamma)}{\twoheadrightarrow} & K(\mathbb{Z}_2, 2) \end{array} \tag{1}$$

Let us consider now the JSS associated to the diagram of the universal example:

$$\begin{array}{ccccc} BSpin & \longrightarrow & BSO & \overset{w}{\twoheadrightarrow} & K(\mathbb{Z}_2, 2) \\ & & \Vert & & \\ & & BSO & & \end{array}$$

Teichner used it to provide another proof of Rohlin's Theorem, and his proof is based on the following observation about the filtration of Ω_4^{SO}:

$$\Omega_4^{Spin} \underbrace{\subseteq}_{\mathbb{Z}_2} F_{2,2} \underbrace{\subseteq}_{\mathbb{Z}_2} F_{3,1} \underbrace{\subseteq}_{\mathbb{Z}_4} \Omega_4^{SO}$$

Since we know that $\mathfrak{sg} \colon \Omega_4^{SO} \xrightarrow{\cong} \mathbb{Z}$, we can infer that $\mathfrak{sg} \colon \Omega_4^{Spin} \to 16 \cdot \mathbb{Z}$ since $2 \cdot 2 \cdot 4 = 16$. in fact one sees that the stable fourth line is the following:

	0	1	2	3	4	5
4	Ω_4^{Spin}					
3		0				
2			$H_2(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$			
1				$H_3(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$		
0					$H_4(K(\mathbb{Z}_2,2);\mathbb{Z})$	

The importance of the universal example is given by the following result:

Proposition 1.0.9. *The filtration of the JSS for*

$$\begin{array}{ccccc} BSpin & \longrightarrow & BSO & \xrightarrow{w_2(\gamma)} & K(\mathbb{Z}_2,2) \\ & & \| & & \\ & & BSO & & \end{array}$$

is characterized by the signature, i.e. $\mathfrak{sg}\colon \Omega_4^{Spin} = F_{0,4} \xrightarrow{\cong} 16{\cdot}\mathbb{Z}$, $\mathfrak{sg}\colon F_{2,2} \xrightarrow{\cong} 8{\cdot}\mathbb{Z}$, $\mathfrak{sg}\colon F_{3,1} \xrightarrow{\cong} 4{\cdot}\mathbb{Z}$ *and* $\mathfrak{sg}\colon \Omega_4^{SO} \xrightarrow{\cong} \mathbb{Z}$

Therefore I was able to retrieve some criteria for finding the divisibility of the signature of a almost spin manifold M with w-type w. Here are two examples:

Corollary 1.0.10. *Let* $\xi\colon B \to BSO$ *be the normal* 1 *type associated to* π *and* $w \neq 0$. *Let* $E_{4,0}^{\infty}$ *be the stable object in the JSS for computing* $\Omega_4(\xi)$. *If* $w_*\colon E_{4,0}^{\infty} \to H_4(K(\mathbb{Z}_2,2);\mathbb{Z})$ *is the zero map then* $4 \mid \mathfrak{sg}(\pi, w)$

Similarly we have

Corollary 1.0.11. *In the above setting, if* $E_{3,1}^{\infty} = 0$ *then* $8 = \mathfrak{sg}(\pi, w)$ *if and only if* $w_*\colon E_{4,0}^{\infty} \to H_4(K(\mathbb{Z}_2,2);\mathbb{Z})$ *is the zero map.*

There are other bordism invariants one can retrieve from the JSS, one of them is the so called *secondary bordism invariant*, which is a map defined on elements M in $\Omega_4(\xi)$ whose signature is zero and has image in the stable object $E_{3,1}^{\infty}$ of the related JSS. Roughly speaking it is the obstruction to extending the ξ-structure of M to the oriented 5-manifold W such that $\partial W = M$, which exists since the signature of M is zero. You can find a precise description of such invariant in [4], page 37.

2 Stable classification for the fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and Dihedral groups

To set the notation once and for all, recall the following results:

Theorem 2.0.1. *we have the following isomorphism:*

- $H^*(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[a, b]$
- *Define* $\omega := w_2(\rho_D)$ *where* $\rho_D \colon D_{2n} \to O(2)$ *is the usual planar representation. Then the following assertions hold:*
  - *If* n *is odd then* $H^*(D_{2n}; \mathbb{Z}_2) \cong \mathbb{Z}_2[y^1]$
  - *If* $n = 2 \pmod 4$ *then* $H^*(D_{2n}; \mathbb{Z}_2) \cong \mathbb{Z}_2[x^1, y^1]$
  - *if* $n = 0 \pmod 4$ *then* $H^*(D_{2n}; \mathbb{Z}_2) \cong \mathbb{Z}_2[x^1, y^1, \omega] \big/ \langle x^1 y^1 + x^2 \rangle$

These are the results I was able to prove so far with the machinery explained above:

Theorem 2.0.2. *Let* M, N *be two closed oriented smooth* 4*-manifolds with fundamental groups isomorphic to* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$*. Let* $c \colon M \to K(\mathbb{Z}_2 \oplus \mathbb{Z}_2, 1)$ *be the map classifying the universal cover. We have the following criterion to detect if they are stable diffeomorphic:*

- *if* M, N *are both totally non spin, then* $M \cong_s N$ *if and only if they have the same signature and the elements* $c_*[M], c_*[N] \in H_4(K(\mathbb{Z}_2 \oplus \mathbb{Z}_2, 1); \mathbb{Z})$ *are both zero or both non-zero.*
- *if* M, N *are both spin, then* $M \cong_s N$ *if and only if they have the same signature and the elements* $c_*[M], c_*[N] \in H_4(K(\Gamma, 1); \mathbb{Z})$ *are both zero or both non-zero.*
- *if* M, N *are both almost spin, then* $M \cong_s N$ *if and only if* $w_M = w_N$*, they have the same signature and in the case that the* w*-type of* M *is* $a^2 \in H^2(K(\mathbb{Z}_2 \oplus \mathbb{Z}_2, 1); \mathbb{Z}_2)$*,* $c_*[M], c_*[N]$ *are both zero or non zero.*

Corollary 2.0.3. *If* M *is an almost spin closed oriented smooth* 4*-manifold with fundamental group* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *the signature is a multiple of:*

- 8 *if* $w_M = a^2, ab^2, \omega$
- 4 *if* $w_M = a^2 + ab + b^2$

Theorem 2.0.4 (Stable Classification for n odd)**.** *Let* M, N *be two closed oriented smooth* 4*-manifolds with fundamental groups isomorphic to* D_{2n} *with* n *odd. We have the following criteria to detect if they are stable diffeomorphic:*

- *if* M, N *are both totally non spin, then* $M \cong_s N$ *if and only if they have the same signature*
- *if* M, N *are both spin, then* $M \cong_s N$ *if and only if they have the same signature*
- *if* M, N *are both almost spin* $M \cong_s N$ *if and only if they have the same signature*

in other words, spin behaviour together with the signature is a complete stable diffeomorphism invariant.

Proof. This is an application of Theorem 1.0.7 together with all the computations done so far in the case for n odd. □

Corollary 2.0.5. *If* M *is an almost spin closed oriented smooth* 4*-manifold with fundamental group* D_{2n}*,* n *odd, its signature is a multiple of* 8*.*

Theorem 2.0.6. *Let* M, N *be two closed oriented smooth* 4*-manifolds with fundamental groups isomorphic to* D_{2n} *with* $n = 2 \pmod 4$*. We have the following criteria to detect if they are stable diffeomorphic:*

- *if M,N are both totally non spin, then $M \cong_s N$ if and only if they have the same signature and one of the following occurs: $(c_M)_*[M] = (c_N)_*[N] = 0$, $(c_M)_*[M] = (c_N)_*[N] = x_3y_1$ or $(c_M)_*[M], (c_N)_*[N] \in \{x_1y_3, x_3y_1 + x_1y_3\}$.*
- *if M,N are both spin, then $M \cong_s N$ if and only if they have the same signature and one of the following occurs: $(c_M)_*[M] = (c_N)_*[N] = 0$, $(c_M)_*[M] = (c_N)_*[N] = x_3y_1$ or $(c_M)_*[M], (c_N)_*[N] \in \{x_1y_3, x_3y_1 + x_1y_3\}$.*
- *if M,N are both almost spin then $M \cong_s N$ if and only if they have the same w-type and one of the following occurs:*
  - *if $w_M = x^2$ and $(c_M)_*[M] = (c_N)_*[N] = 0$ or $(c_M)_*[M] = (c_N)_*[N] = x_3y_1$ or $(c_M)_*[M], (c_N)_*[N] \in \{x_1y_3, x_3y_1 + x_1y_3\}$ and they have the same signature*
  - *if $w_M = y^2$ and $(c_M)_*[M] = (c_N)_*[N] = 0$ or $(c_M)_*[M] = (c_N)_*[N] = x_3y_1$ or $(c_M)_*[M], (c_N)_*[N] \in \{x_1y_3, x_3y_1 + x_1y_3\}$ and they have the same signature*
  - *if $w_M = x^1y^1$ and they have the same signature*
  - *if $w_M = x^2 + x^1y^1$ and they have the same signature*
  - *if $w_M = x^2 + x^1y^1 + y^2$ and they have the same signature*

Proof. This is an application of Theorem 1.0.7 together with all the computations done so far in the case for $n = 2 \pmod 4$. □

Corollary 2.0.7. *If M is an almost spin closed oriented smooth 4-manifold with fundamental group D_{2n}, $n = 2 \pmod 4$ the signature is a multiple of:*

- 8 *if $w_M = x^2, y^2, x^1y^1, x^2 + x^1y^1$*
- 4 *if $w_M = x^2 + x^1y^1 + y^2$*

Much more interesting (and harder) is the case of the dihedral group D_{2n} with $n = 0 \pmod 4$. I still need to deal with the case $w_2M = y^2$

Theorem 2.0.8 (Stable Classification for $n = 0 \pmod 4$)**.** *Let M,N be two closed oriented smooth 4-manifolds with fundamental groups isomorphic to D_{2n} with $n = 0 \pmod 4$. We have the following criteria to detect if they are stable diffeomorphic:*

- *if M,N are both totally non spin, then $M \cong_s N$ if and only if they have the same signature and $(c_M)_*[M] = (c_N)_*[N] = 0$, or $(c_M)_*[M] = (c_N)_*[N] = \omega x_2$, or $(c_M)_*[M], (c_N)_*[N] \in \{\omega y_2, \omega y_2 + \omega x_2\}$*
- *if M,N are both spin, then $M \cong_s N$ if and only if they have the same signature and $(c_M)_*[M] = (c_N)_*[N] = 0$, or $(c_M)_*[M] = (c_N)_*[N] = \omega x_2$, or $(c_M)_*[M], (c_N)_*[N] \in \{\omega y_2, \omega y_2 + \omega x_2\}$*
- *if M,N are both almost spin then $M \cong_s N$ if and only if they have the same w-type and one of the following occurs:*
  - *if $w_M = x^2$ and they have the same signature and $(c_M)_*[M], (c_N)_*[N]$ are both zero or non-zero (This might be not an optimal characterization!).*
  - *if $w_M = y^2$*
  - *if $w_M = \omega$ and they have the same signature.*
  - *if $w_M = \omega + x^2$ and they have the same signature.*
  - *if $w_M = \omega + y^2$ and they have the same signature.*

Corollary 2.0.9. *If M is an almost spin closed oriented smooth 4-manifold with fundamental group D_{2n}, $n = 0 \pmod 4$ the signature is a multiple of:*

- 8 *if* $w_M = x^2, y^2, \omega$
- 4 *if* $w_M = \omega + x^2, \omega + y^2$

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